Reflection of stationary sets and GDST

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Young Set Theory Workshop 2022

20 August, 2022

This is a joint work with Gabriel Fernandes and Assaf Rinot.

Generalised Baire space

Suppose κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

The generalised Baire space is the space κ^κ endowed with the bounded topology, for every $\eta \in \kappa^{<\kappa}$ the following set

$$N_\eta = \{\xi \in \kappa^\kappa \mid \eta \subseteq \xi\}$$

is a basic open set.

The generalised Cantor space is the subspace 2^κ.

Reductions

For $i < 2$, let X_i be some space from the collection $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$. Let R_0 and R_1 be binary relations over X_0 and X_1, respectively.

Definition

A function $f : X_0 \to X_1$ is said to be a *reduction of* R_0 *to* R_1 iff, for all $\eta, \xi \in X_0$,

$$\eta \; R_0 \; \xi \text{ iff } f(\eta) \; R_1 \; f(\xi).$$

The existence of a function f satisfying this is denoted by $R_0 \hookrightarrow R_1$.

Complexity

The existence of a continuous reduction f is denoted by $R_0 \hookrightarrow_c R_1$. We say that R_0 is at most as complex as R_1.

In the case $R_0 \hookrightarrow_c R_1$ and $R_1 \not\hookrightarrow_c R_0$, we say that R_0 is less complex than R_0.

In generalized descriptive set theory, the complexity of a theory can be study by studying the complexity of the isomorphism relation of the theory.

Question

In model theory, the complexity of a theory can be measure by the number of models of the theory.

Shelah's main gap theorem can be understood as: *Classifiable theories are less complex than non-classifiable theories, in the model theory complexity.*

Question. Are classifiable theories less complex than non-classifiable theories, in the generalised descriptive set theory complexity?

Stationary reflection

Let α be an ordinal of uncountable cofinality. A set $C \subseteq \alpha$ is a club if it is closed and unbounded. A set $S \subseteq \alpha$ is stationary if for all club $C \subset \alpha$, $C \cap S \neq \emptyset$.

Definition

Let $\alpha \in \kappa$ be an ordinal of uncountable cofinality, and a stationary $S \subseteq \kappa$, we say that S reflects at α if $S \cap \alpha$ is stationary in α

If κ is a weakly compact cardinal, every stationary subset of κ reflects at a regular cardinal $\alpha < \kappa$.

Equivalence modulo nonstationary

Definition

For every stationary set $S \subseteq \kappa$ and $\theta \in [2, \kappa]$, the equivalence relation $=^\theta_S$ over the subspace θ^κ is defined via

$$\eta =^\theta_S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\} \text{ is non-stationary.}$$

Definition

The quasi-order $\leq^S$ over κ^κ is defined via

$$\eta \leq^S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \text{ is non-stationary.}$$

The quasi-order $\subseteq^S$ over 2^κ is nothing but $\leq^S \cap (2^\kappa \times 2^\kappa)$.

Progress

Let us denote by $=^{\theta}_{\lambda}$ the relation $=^{\theta}_{S}$ when $S = \{\alpha < \kappa \mid cf(\alpha) = \lambda\}$.

Fact (Hyttinen-M)

The isomorphism relation of any classifiable theory is less complex than $=^{\kappa}_{\lambda}$ for all λ.

Under some cardinal arithmetic assumptions the following can be proved:

Fact (Friedman-Hyttinen-Kulikov)

Suppose T is a non-classifiable theory. There is a regular cardinal $\lambda < \kappa$ such that $=^{2}_{\lambda}$ is at most as complex as the isomorphism relation of T.

Question

Is $=^\kappa_\lambda$ Borel-reducible to $=^2_\lambda$, i.e. $=^\kappa_\lambda \hookrightarrow_c =^2_\lambda$, for all λ?

Comparing $=^\kappa_S$ and $=^2_S$

Fact (Asperó-Hyttinen-Kulikov-M)

If every stationary subset of X *reflects at stationary many* $\alpha \in Y$*, then* $=^\kappa_X \hookrightarrow_c =^\kappa_Y$.

Fact (Friedman-Hyttinen-Kulikov)

Suppose $V = L$*, and* $X \subseteq \kappa$ *and* $Y \subseteq reg(\kappa)$ *are stationary. If every stationary subset of* X *reflects at stationary many* $\alpha \in Y$*, then* $=^2_X \hookrightarrow_c =^2_Y$.

Limitations

- For all regular cardinals $\gamma \leq \lambda < \kappa$, any $X \subseteq S^\kappa_\lambda$, X does not reflect at any $\alpha \in S^\kappa_\gamma$.
- If $\kappa = \lambda^+$ and $\square_\lambda$ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that Y does not reflect at any $\alpha < \kappa$.
- Usual stationary reflection requires large cardinals.

Capturing clubs

Suppose S is stationary subset of κ, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α.

Definition

We say that $\vec{\mathcal{F}}$ *captures clubs* iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is non-stationary;

For any ordinal $\alpha < \kappa$ of uncountable cofinality, denote by $CUB(\alpha)$ the club filter of subsets of α. The sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S^\kappa_{\omega_1} \rangle$ define by $\mathcal{F}_\alpha = CUB(\alpha)$, capture clubs.

Filter reflection

Suppose X and S are stationary subsets of κ, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α.

Definition

We say that X $\vec{\mathcal{F}}$-reflects to S iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary

Definition

We say that X $\mathfrak{f}$-*reflects to* S iff there exists a sequence of filters $\vec{\mathcal{F}}$ over a stationary subset S' of S such that X $\vec{\mathcal{F}}$-reflects to S'.

We refer as fake reflection to the case when $\mathcal{F}_\alpha \neq CUB(\alpha)$ for all α.

Strong forms of filter reflection

Definition

We say that X strongly $\vec{\mathcal{F}}$-reflects to S iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha\}$ is stationary.

Definition

We say that X *$\vec{\mathcal{F}}$-reflects with $\diamondsuit$ to S* iff $\vec{\mathcal{F}}$ captures clubs and there exists a sequence $\langle Y_\alpha \mid \alpha \in S\rangle$ such that, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y_\alpha = Y \cap \alpha \ \&\ Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary.

We apply the same convention for X *strongly $\mathfrak{f}$-reflects to S* and X *$\mathfrak{f}$-reflects with $\diamondsuit$ to S*

Properties

Fact (Monotonicity)

For stationary sets $Y \subseteq X \subseteq \kappa$ and $S \subseteq T \subseteq \kappa$. If X $\mathfrak{f}$-reflects to S, then Y $\mathfrak{f}$-reflects to T;

Fact

Suppose X strongly $\mathfrak{f}$-reflects to S. If $\diamondsuit_X$ holds, then so does $\diamondsuit_S$.

Fact

Suppose $V = L$, then for all stationary sets $X, S \subseteq \kappa$, X $\mathfrak{f}$-reflects to S.

Over the limits

- Usual stationary reflection is a special case of filter reflection.
- For all regular cardinals $\gamma \leq \lambda < \kappa$, any $X \subseteq S^\kappa_\lambda$, X does not reflect at any $\alpha \in S^\kappa_\gamma$. S^κ_λ $\mathfrak{f}$-reflects to S^κ_γ is consistently true.
- If $\kappa = \lambda^+$ and $\square_\lambda$ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that Y does not reflect at any $\alpha < \kappa$. Fake reflection is consistent with $\square_\lambda$.
- Fake reflection does not require large cardinals.

Killing Filter Reflection

Theorem

There exists a cofinality-preserving forcing extension in which, for all stationary subsets X, S of κ, X does not $\mathfrak{f}$-reflect to S

Force a coherent regressive C-sequence, then force with $Add(\kappa, \kappa^+)$.

Forcing Filter Reflection

Theorem

For all stationary subsets X and S of κ, there exists a $<\kappa$-closed κ^+-cc forcing extension, in which X $\mathfrak{f}$-reflects to S.

Force with Sakai's forcing

Stationary Reflection

Theorem

If κ is strongly inaccessible, then in the forcing extension by $Add(\kappa, \kappa^+)$, for all two disjoint stationary subsets X, S of κ, the following are equivalent:

1. *X $\mathfrak{f}$-reflects to S;*
2. *every stationary subset of X reflects in S.*

$Add(\omega, \kappa)$

Theorem

Suppose X $\mathfrak{f}$-reflects to S holds. Then $Add(\omega, \kappa)$ forces that X $\mathfrak{f}$-reflects to S.

Preserving Filter Reflection

Definition

Let $\kappa = \lambda^+$. A notion of forcing $\mathbb{Q}$ satisfies κ-stationary-cc if for every sequence $\langle q_\delta \mid \delta < \kappa \rangle$ of conditions in $\mathbb{Q}$ there is a club $D \subseteq \kappa$ and a regressive map $h : D \cap E^\kappa_{\mathrm{cof}(\lambda)} \to \kappa$ such that for all $\gamma, \delta \in dom(h)$, if $h(\gamma) = h(\delta)$ then q_γ and q_δ are comparable.

Definition

We say that a sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is *θ-complete* if the set $\{\alpha \in S \mid \mathcal{F}_\alpha \textit{ is not } \theta\textit{-complete}\}$ is non-stationary.

Preserving Filter Reflection

Theorem

Suppose $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a θ-complete sequence. Suppose $\mathbb{P}$ is a forcing notion with θ-cc and κ-stationary-cc, and $X \subseteq \kappa$ is a stationary set such that X $\vec{\mathcal{F}}$-reflects to S. Then $\mathbb{P}$ forces that X $\vec{\mathfrak{f}}$-reflects to S.

Equivalence Modulo a Filter

Definition

Let $\mathcal{F}$ be a filter over α. For every $\theta \in [2, \kappa]$, the equivalence modulo $\mathcal{F}$, $\sim^{\theta}_{\mathcal{F}}$, over θ^{α}, is defined via

$$(\eta, \xi) \ \in \sim^{\theta}_{\mathcal{F}} \ \text{iff } \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\} \in \mathcal{F}$$

Definition

For every $\theta, \gamma \in [2, \kappa]$, $F : \theta^{\kappa} \to \gamma^{\kappa}$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_{\alpha} \mid \alpha \in S \rangle$ a sequence of filters. We say that F captures $\vec{\mathcal{F}}$ if for all $\alpha \in S$ and $\eta, \xi \in \theta^{\kappa}$

$$\eta \restriction \alpha \ \sim^{\theta}_{\mathcal{F}_{\alpha}} \ \xi \restriction \alpha \text{ iff } F(\eta)(\alpha) = F(\xi)(\alpha).$$

Characterization of Filter Reflection

Theorem

Let $X, S \subseteq \kappa$ be stationary sets. The following are equivalent:

1. *X $\mathfrak{f}$-reflects to S.*
2. *There is $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ and $F : 2^\kappa \to \kappa^\kappa$ a reduction from $=^2_X$ to $=^\kappa_S$ that captures $\vec{\mathcal{F}}$.*

Characterization of Strong Filter Reflection

Theorem

Let $X, S \subseteq \kappa$ be stationary sets. The following are equivalent.

1. *X strongly $\mathfrak{f}$-reflects to S.*
2. *There is a reduction $F : 2^\kappa \to 2^\kappa$ from $=^2_X$ to $=^2_S$ such that for all $\alpha \in S$ the set*

$$\{\eta \restriction_\alpha^{-1} [\gamma] \mid \eta \in 2^\kappa \;\&\; F(\eta)(\alpha) = \gamma\}$$

is a filter.

Reductions from Filter Reflection

Lemma

If X strongly $\mathfrak{f}$-reflects to S, then for all $\theta \in [2, \kappa]$, $=^{\theta}_{X} \hookrightarrow_{c} =^{\theta}_{S}$.

Theorem

Let $X, S \subseteq \kappa$ be stationary sets. If X $\mathfrak{f}$-reflects with $\diamondsuit$ to S, then

$$\leq^{X} \hookrightarrow_{c} \subseteq^{S}.$$

The Main Gap

Theorem

Let $\kappa = \lambda^+$. If cof(ω) $\mathfrak{f}$-reflects with $\diamondsuit$ to cof(ω) and cof(λ) $\mathfrak{f}$-reflects with $\diamondsuit$ to cof(λ), then the isomorphism relation of any classifiable theory is continuous reducible to the isomorphism relation of any non-classifiable theory.

The Main Gap

Theorem

Let $\kappa = \lambda^+$ *and* $X \subseteq \kappa$ *a stationary set such that* $X \cap cof(\lambda) = \emptyset$. *If* X *strongly* $\mathfrak{f}$*-reflects to* $cof(\lambda)$, *then the isomorphism relation of any classifiable theory is continuous reducible to the isomorphism relation of any non-classifiable theory.*

Martin Maximum

Theorem

Suppose Martin's Maximum holds, $\kappa > \omega_2$, $X \subseteq cof(\omega)$ a stationary set and $S \subseteq cof(\omega_1)$. If $\diamondsuit_X$ holds, then X reflects with $\diamondsuit$ to S.

MM implies that for $\kappa = \lambda^+$, λ a singular strong limit of uncountable cofinality, it holds

$$=^{\kappa}_{\omega} \hookrightarrow_c =^{2}_{\omega_1}.$$

Remains Open

Is the following consistently true?
For all $S \subseteq \kappa$ stationary,

$$=^{\kappa}_{S} \not\hookrightarrow_{c} =^{2}_{S}.$$

Thank you