A Borel-reducibility Main Gap

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Geometry

- Independence of Euclid's fifth postulate, the parallel postulate.
- Khayyám (1077) and Saccheri (1733) considered the three different cases of the Khayyám-Saccheri quadrilateral (right, obtuse, and acute).
- Euclidean geometry, Elliptic geometry, Hyperbolic geometry.

The spectrum fuction

Let T be a countable theory over a countable language. Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T,\alpha)$?

Categoricity

- **1904:** Veble introduced categorical theories.
- **1915 - 1920:** Löwenheim-Skolem Theorem.
- **1929:** Gödel's completeness theorem.
- **1954:** Łoś and Vaught introduced κ-categorical theories.
- **1965:** Morley's categoricity theorem.

Morley's conjecture

1960's: Let T be a first-order countable theory over a countable language. For all $\aleph_0 < \lambda < \kappa$,

$$I(T,\lambda) \leq I(T,\kappa).$$

1990: Shelah proved Morley's conjecture.

Shelah's Main Gap Theorem

Theorem (Shelah 1990)

Either, for every uncountable cardinal α, $I(T,\alpha)=2^\alpha$; or $\forall\alpha>0$, $I(T,\aleph_\alpha)<\beth_{\omega_1}(|\alpha|)$.

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Descriptive Set Theory

- **1989:** Friedman and Stanley introduced the Borel reducibility between classes of countable structures.
- **1993:** Mekler-Väänänen κ-separation theorem.
- **2014:** Friedman-Hyttinen-Kulikov developed GDST and a systematic comparison between the Main Gap dividing lines and the complexity given by Borel reducibility.

The bounded topology

Let κ be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

The Generalised Baire spaces

The generalised Baire space is the space κ^κ endowed with the bounded topology.

The generalised Cantor space is the subspace 2^κ.

Coding structures

Let $\omega \leq \mu \leq \kappa$ be a cardinal. Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$ and a bijection π_μ between $\mu^{<\omega}$ and μ.

Definition

For every $\eta \in \kappa^\kappa$ define the structure $\mathcal{A}_{\eta\restriction\mu}$ with domain μ as follows: For every tuple $(a_1, a_2, \ldots, a_n)$ in μ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_{\eta\restriction\mu}} \Leftrightarrow \eta(\pi_\mu(m, a_1, a_2, \ldots, a_n)) > 0.$$

The isomorphism relation

Definition

Let $\omega \leq \mu \leq \kappa$ be a cardinal and T a first-order theory in a relational countable language, we say that $f, g \in \kappa^\kappa$ are $\cong^\mu_T$ equivalent if one of the following holds:

- $\mathcal{A}_{\eta\restriction\mu} \models T, \mathcal{A}_{\xi\restriction\mu} \models T, \mathcal{A}_{\eta\restriction\mu} \cong \mathcal{A}_{\xi\restriction\mu}$
- $\mathcal{A}_{\eta\restriction\mu} \not\models T, \mathcal{A}_{\xi\restriction\mu} \not\models T$

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *reducible* to E_2, if there is a function $f: \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$. We write $E_1 \hookrightarrow_r E_2$.

We can define a partial order on the set of all first-order complete countable theories

$$T \leq^\kappa T' \textit{ iff } \cong_T \hookrightarrow_C \cong_{T'}$$

Non-classifiable theories

A theory T is non-classifiable if it is a countable complete theory that satisfies one of the following:

- T is unstable;
- T is stable unsuperstable;
- T is superstable with DOP; %pause
- T is superstable with OTOP.

Classifiable theories

Classifiable are divided into:

- shallow,

$$I(T,\aleph_\alpha) < \beth_{\omega_1}(|\ \alpha\ |);$$

- non-shallow,

$$I(T,\alpha) = 2^\alpha.$$

First dividing lines

Fact (Friedman-Hyttinen-Kulikov 2014)

1. *Let $\kappa^{<\kappa} = \kappa > 2^\omega$. If T is classifiable and shallow, then $\cong_T$ is κ-Borel.*
2. *If T is classifiable non-shallow, then $\cong_T$ is $\Delta_1^1(\kappa)$ not κ-Borel.*
3. *If T is unstable or stable with the OTOP or superstable with the DOP and $\kappa > \omega_1$, then $\cong_T$ is not $\Delta_1^1(\kappa)$.*
4. *If T is stable unsuperstable, then $\cong_T$ is not κ-Borel.*

Question

Question: What can we say about the Borel-reducibility between different dividing lines?

Classifiable and shallow

Theorem (Mangraviti - Motto Ros 2020)

Let κ be such that $\kappa > 2^\omega$. If T is classifiable and shallow with depth α, then $rk_B(\cong_T) \leq 4\alpha$.

Theorem (Mangraviti - Motto Ros 2020)

Let $\kappa = \aleph_\gamma$ be such that $\kappa^{<\kappa} = \kappa$ and $\beth_{\omega_1}(|\gamma|) \leq \kappa$. Let T, T' be countable complete first-order theories, and suppose T is classifiable and shallow, while T' is not. Then

$$\cong_T \hookrightarrow_B \cong_{T'}$$

General reduction

Fact (Mangraviti-Motto Ros)

Let E_1 be a Borel equivalence relation with $\gamma \leq \kappa$ equivalence classes and E_2 be an equivalence relation with θ equivalence classes. If $\gamma \leq \theta$, then $E_1 \hookrightarrow_B E_2$.

1_ϱ relation

Let $0 < \varrho \leq \kappa$. $\eta \; 1_\varrho \; \xi$ if and only if one of the following holds:

- ϱ is finite:
  - $\eta(0) = \xi(0) < \varrho - 1$;
  - $\eta(0), \xi(0) \geq \varrho - 1$.
- ϱ is infinite:
  - $\eta(0) = \xi(0) < \varrho$;
  - $\eta(0), \xi(0) \geq \varrho$.

Few equivalence classes

Lemma (M. 2023)

Suppose $\kappa > 2^\omega$ *and* T *is a countable first-order theory in a countable vocabulary (not necessarily complete) such that* $\cong_T$ *has* $\varrho \leq \kappa$ *equivalence classes. Then*

$$\cong_T \hookrightarrow_B 1_\varrho \text{ and } 1_\varrho \hookrightarrow_L \cong_T .$$

Even more, if T *is not categorical then* $\cong_T \not\hookrightarrow_C 1_\varrho$.

Proof

- $\cong_T \hookrightarrow_B 1_\varrho$ follows from Mangraviti-Motto Ros.
- $\eta \restriction 1$ determines the equivalence class of η. So $1_\varrho \hookrightarrow_L \cong_T$.
- 1_ϱ is open, so $\cong_T \hookrightarrow_C 1_\varrho$ implies $\cong_T$ is open.
- $\cong_T$ is open iff T is categorical (Mangraviti-Motto Ros), so if T is not categorical then $\cong_T \not\hookrightarrow_C 1_\varrho$.

Gap: Shallow and Non-shallow

Theorem (M. 2023)

Suppose $\aleph_\mu = \kappa = \lambda^+ = 2^\lambda$ is such that $\beth_{\omega_1}(|\mu|) \leq \kappa$. Let T_1 be a countable complete classifiable shallow theory with $\varrho = I(\kappa, T_1)$, T_2 be a countable complete theory not classifiable shallow. If T is classifiable shallow such that $1 < I(\kappa, T) < I(\kappa, T_1)$, then

$$\cong_T \hookrightarrow_B 1_\varrho \hookrightarrow_L \cong_{T_1} \hookrightarrow_B 1_\kappa \hookrightarrow_L \cong_{T_2}.$$

In particular

$$\cong_{T_2} \not\hookrightarrow_r 1_\kappa \not\hookrightarrow_r \cong_{T_1} \not\hookrightarrow_C 1_\varrho \not\hookrightarrow_r \cong_T.$$

Consistency

Theorem (Hyttinen - Kulikov - M. 2017)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, *and* $\lambda^{<\lambda} = \lambda$. *There is a* κ*-closed* κ^+*-cc forcing which forces: If* T *is classifiable and* T' *is non-classifiable, then* $T \leq^\kappa T'$ *and* $T' \not\leq^\kappa T$.

Unsuperstable theories

Theorem (Hyttinen - Kulikov - M. 2017)

Suppose $\kappa=\lambda^+$, $2^\lambda>2^\omega$, and $\lambda^\omega=\lambda$. If T is classifiable and T' is stable unsuperstable, then $T\leq^\kappa T'$ and $T'\not\leq^\kappa T$.

Theorem (M. 2022)

Suppose $\kappa=\lambda^+=2^\lambda$ and $\lambda^\omega=\lambda$. If T is a classifiable theory, and T' is an unsuperstable theory, then $T\leq^\kappa T'$ and $T'\not\leq^\kappa T$.

Equivalence modulo γ cofinality

Definition

We define the equivalence relation $=^2_\gamma \subseteq 2^\kappa \times 2^\kappa$, as follows: let $S = \{\alpha < \kappa \mid cf(\alpha) = \gamma\}$,

$$\eta =^2_\gamma \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

Borel-reducibility Main Gap

Theorem (M. 2023)

Let $\mathfrak{c} = 2^\omega$. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $2^\mathfrak{c} \leq \lambda = \lambda^{\omega_1}$. If T is a classifiable theory, and T' is a non-classifiable theory, then there is $\gamma < \kappa$ such that

$$\cong_T \hookrightarrow_C =^2_\gamma \hookrightarrow_C \cong_{T'} \text{ and } =^2_\gamma \not\hookrightarrow_B \cong_T .$$

In particular

$$T \ \leq^\kappa \ T' \text{ and } T' \ \not\leq^\kappa \ T.$$

Classifiable theories

Theorem (Hyttinen - Kulikov - M. 2017)

Assume T is a classifiable theory. If $\diamondsuit_S$ holds, then $\cong_T \hookrightarrow_C =^2_\gamma$.

The reductions

Theorem (M. 2023)

Let κ be inaccessible or $\kappa = \lambda^+ = 2^\lambda$. Suppose T is a non-classifiable theory.

1. *If T is stable unsuperstable, then let $\theta = \gamma = \omega$.*
2. *If T is unstable, or superstable with OTOP, then let $\theta = \omega$ and $\omega \leq \gamma < \kappa$.*
3. *If T is superstable with DOP, then let $\theta = 2^\omega = \mathfrak{c}$ and $\omega_1 \leq \gamma < \kappa$.*

If θ, γ, and κ satisfy that $\forall \alpha < \kappa$, $\alpha^\gamma < \kappa$, and $(2^\theta)^+ \leq \kappa$, then

$$=^2_\gamma \hookrightarrow_C \cong_T .$$

Ordered trees

Definition

Let $\gamma < \kappa$ be a regular cardinal and I a linear order. $(A, \prec, <)$ is an ordered tree if the following holds:

- $(A, \prec)$ is a κ^+, $(\gamma + 2)$-tree*.
- for all $x \in A$, $(succ(x), <)$ is isomorphic to I.

Isomorphism of trees

Theorem (M. 2023)

Suppose $\gamma < \kappa$ is such that for all $\epsilon < \kappa$, $\epsilon^\gamma < \kappa$, and there is a κ-colorable linear order I. For all $f \in 2^\kappa$ there is an ordered tree A_f such that for all $f, g \in 2^\kappa$,

$$f \ =^2_\gamma \ g \Leftrightarrow A_f \cong A_g.$$

The models

Lemma (M. 2023)

Suppose T is unstable or superstable with DOP or OTOP in a countable relational vocabulary τ. If A is an ordered tree with $(succ(x), <)$ is ω_1-dense, then there is an Ehrenfeucht-Mostowski model, $\mathcal{M}(A)$, with the skeleton indiscernible in $\mathcal{M}(A)$ relative to $L_{\infty\omega_1}$.

The isomorphism theorem

Theorem (M. 2023)

Suppose T is unstable or superstable with DOP or OTOP in a countable relational vocabulary τ. If there is a ω_1-dense, $(\kappa, bs, bs, \omega_1)$-nice, $(<\kappa, bs)$-stable, and κ-colorable linear order, then for all $f, g \in 2^\kappa$,

$$f \ =^2_\gamma \ g \text{ iff } \mathcal{M}(A_f) \cong \mathcal{M}(A_g).$$

ε-dense

Definition

Let I be a linear order of size κ and ε a regular cardinal smaller than κ. We say that I is *ε-dense* if the following holds.

If $A, B \subseteq I$ are subsets of size less than ε such that for all $a \in A$ and $b \in B$, $a < b$, then there is $c \in I$, such that for all $a \in A$ and $b \in B$, $a < c < b$.

κ-representation

Definition

Let A be an arbitrary set of size κ. The sequence $\mathbb{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ is a *κ-representation* of A, if $\langle A_\alpha \mid \alpha < \kappa \rangle$ is an increasing continuous sequence of subsets of A, for all $\alpha < \kappa$, $|A_\alpha| < \kappa$, and $\bigcup_{\alpha<\kappa} A_\alpha = A$.

$(\kappa, bs, bs, \varepsilon)$-nice

Definition

Let $\varepsilon < \kappa$ be a regular cardinal, A be a linear order of size κ and $\langle A_\alpha \mid \alpha < \kappa \rangle$ a κ-representation. Then A is $(\kappa, bs, bs, \varepsilon)$-*nice* if there is a club $C \subseteq \kappa$, such that for all limit $\delta \in C$ with $cf(\delta) \geq \varepsilon$, for all $x \in A$ there is $\beta < \delta$ such that one of the following holds:

- $\forall \sigma \in A_\delta[\sigma \geq x \Rightarrow \exists \sigma' \in A_\beta \ (\sigma \geq \sigma' \geq x)]$
- $\forall \sigma \in A_\delta[\sigma \leq x \Rightarrow \exists \sigma' \in A_\beta \ (\sigma \leq \sigma' \leq x)]$

$(<\kappa, bs)$-stable

Definition

A linear order I is $(<\kappa, bs)$-*stable* if for every $B \subseteq I$ of size smaller than κ,

$$\kappa > |\{tp_{bs}(a, B, I) \mid a \in I\}|.$$

κ-colorable

Definition

Let I be a linear order of size κ. We say that I is *κ-colorable* if there is a function $F : I \to \kappa$ such that for all $B \subseteq I$, $|B| < \kappa$, $b \in I \backslash B$, and $p = tp_{bs}(b, B, I)$ such that the following hold: For all $\alpha \in \kappa$,

$$|\{a \in I \mid a \models p \ \& \ F(a) = \alpha\}| = \kappa.$$

Existence

Let $\theta < \kappa$ be the smallest cardinal such that there is a ε-dense model of *DLO* of size θ.

Theorem (M. 2023)

Suppose κ is inaccessible, or $\kappa = \lambda^+$, $2^\theta \leq \lambda = \lambda^{<\varepsilon}$. There is a ε-dense, $(\kappa, bs, bs, \varepsilon)$-nice, $(<\kappa, bs)$-stable, and κ-colorable linear order.

Construction

Let $\mathcal{Q}$ be a model of *DLO* of size $\theta < \kappa$, that is ε-dense.

Definition

Let $\kappa \times \mathcal{Q}$ be ordered by the lexicographic order, I^0 be the set of functions $f : \varepsilon \to \kappa \times \mathcal{Q}$ such that $f(\alpha) = (f_1(\alpha), f_2(\alpha))$, for which $|\{\alpha \in \varepsilon \mid f_1(\alpha) \neq 0\}|$ is smaller than ε.
If $f, g \in I^0$, then $f < g$ if and only if $f(\alpha) < g(\alpha)$, where α is the least number such that $f(\alpha) \neq g(\alpha)$.

Construction

Let us fix $\tau \in \mathcal{Q}$. Let I be the set of functions $f : \varepsilon \to (\{0\} \times I^0) \cup (\kappa \times \mathcal{Q})$ such that the following hold:

- $f \restriction \{0\} : \{0\} \to \{0\} \times I^0$;
- $f \restriction \varepsilon \backslash \{0\} : \varepsilon \backslash \{0\} \to \kappa \times \mathcal{Q}$;
- there is $\alpha < \varepsilon$ ordinal such that $\forall \beta > \alpha$, $f(\beta) = (0, \tau)$. We say that the least α with such property is the *depth* of f and we denote it by $dp(f)$;
- there are functions $f_1 : \varepsilon \to \kappa$ and $f_2 : \varepsilon \to I^0 \cup \mathcal{Q}$ such that $f(\beta) = (f_1(\beta), f_2(\beta))$ and $f_1 \restriction dp(f) + 1$ is strictly increasing.

Construction

We say that $f < g$ if and only if one of the following holds:

- $f(0) \neq g(0)$ and $f_2(0) < g_2(0)$;
- let $\alpha = dp(g)$, $\forall \beta \leq \alpha$, $f(\beta) = g(\beta)$ and $f_1(\alpha + 1) \neq 0$;
- exists $\alpha > 0$ such that $\forall \beta < \alpha$, $f(\beta) = g(\beta)$, and $f_1(\alpha), g_1(\alpha) \neq 0$ and $g(\alpha) > f(\alpha)$.

Generators

Definition

For all $f \in I$ with depth α, define the *generator* of f, $Gen(f)$, by

$$Gen(f) = \{g \in I \mid f \restriction \alpha + 1 = g \restriction \alpha + 1\}.$$

Generators

- If $f \neq g$ and $g \in Gen(f)$, then $f > g$.
- Let $f \in Gen(\nu)$. If $g \notin Gen(\nu)$, then $g < \nu$ if and only if $g < f$.
- If $f \in Gen(\nu)$

$$f \models tp_{bs}(\nu, I \backslash Gen(\nu), I) \cup \{\nu > x\}.$$

- Let $f \in Gen(\nu)$. If $\sigma \in I$ is such that $\nu \geq \sigma \geq f$, then $\sigma \in Gen(\nu)$.

Iterations

For all $f \in I$ with depth α, define $o(f) = f_1(\alpha)$ the *complexity* of f.

Suppose i is such that I^i is defined. Let

$$I^{i+1} = \{f \in I \mid o(f) \leq i + 1\}.$$

Suppose i is a limit ordinal such that for all $j < i$, I^j is defined, let

$$I^i = \bigcup_{j<i} I^j.$$

κ-representation

Define $\langle I^0_\alpha \mid \alpha < \kappa\rangle$ by

$$I^0_\alpha = \{\nu \in I^0 \mid \nu_1(n) < \alpha \textit{ for all } n < \varepsilon\}.$$

Suppose $i < \kappa$ is such that $\langle I^i_\alpha \mid \alpha < \kappa\rangle$ has been defined. For all $\alpha < \kappa$ let

$$I^{i+1}_\alpha = \{f \in I \mid o(f) \leq i+1 \ \& \ f_2(0) \in I^0_\alpha\},$$

for $i < \kappa$ is a limit ordinal so

$$I^i_\alpha = \cup_{j<i} I^j_\alpha.$$

κ-representation

Let us define the κ-representation $\langle I_\alpha \mid \alpha < \kappa\rangle$ by

$$I_\alpha = I^\alpha_\alpha.$$

Let $\nu \in I^i_\delta$. For all $f \in Gen(\nu)$, $f \in I^{o(f)}_\delta$.

Roads

Definition

For all $\nu \in I$ with $dp(\nu) = \alpha$, there is a maximal sequence $\langle \nu_i \mid i \leq \alpha \rangle$ such that $\nu_0 \in I^0$, $\nu_\alpha = \nu$, and for all $i < j$, $\nu_i \in Gen(\nu_i)$.
We call this sequence *the road from I^0 to ν*.

Fact

Let $\langle \nu_j \mid j \leq \alpha \rangle$ be the road from I^0 to ν_α. For all $i < \alpha$

$$\nu_\alpha \models tp_{bs}(\nu_i, I^{o(\nu_{i+1})} \backslash Gen(\nu_{i+1}), I) \cup \{\nu_i > x\}$$

$\cong_T \hookrightarrow_C =^2_\mu$, $\kappa = \lambda^+$

Theory	$\lambda = \lambda^\gamma$	$\diamondsuit_\lambda$	$\mathrm{DI}^*_{S^\kappa_\gamma}(\Pi^1_1)$
Classifiable	$\omega \leq \mu \leq \gamma$	$\mu = \lambda$	$\mu = \gamma$
Non-classifiable	Indep	Indep	$\mu = \gamma$

$=^2_\mu \hookrightarrow_C \cong_T$, $\kappa = \lambda^+$

Theory	$\lambda = \lambda^\gamma$	$2^{\mathfrak{c}} \leq \lambda = \lambda^\gamma$	$2^{\mathfrak{c}} \leq \lambda = \lambda^{<\lambda}$ & $\diamondsuit_\lambda$
Stable Unsuper-stable	$\mu = \omega$	$\mu = \omega$	$\mu = \omega$
Unstable	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \lambda$
Superstable with OTOP	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \lambda$
Superstable with DOP	?	$\omega_1 \leq \mu \leq \gamma$	$\omega_1 \leq \mu \leq \lambda$

A bigger Gap

Theorem (M. 2023)

Suppose κ is inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{\omega_1}$. There exists a cofinality-preserving forcing extension in which the following holds:
If T_1 is classifiable and T_2 is not. Then there is a regular cardinal $\gamma < \kappa$ such that, if $X, Y \subseteq S^\kappa_\gamma$ are stationary and disjoint, then $=^2_X$ and $=^2_Y$ are strictly in between $\cong_{T_1}$ and $\cong_{T_2}$.

Main Gap Dichotomy

Theorem (M. 2023)

Let κ be inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. There exists a $<\kappa$-closed κ^+-cc forcing extension in which for any countable first-order theory in a countable vocabulary (not necessarily complete), T, one of the following holds:

- $\cong_T$ *is* $\Delta_1^1(\kappa)$;
- $\cong_T$ *is* $\Sigma_1^1(\kappa)$*-complete.*

Non-classifiable theories

Lemma (M. 2023)

Let κ be strongly inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. For all cardinals $\aleph_0 < \mu < \delta < \kappa$, if T is a non-classifiable theory then

$$\cong^{\mu}_{T} \hookrightarrow_C \cong^{\delta}_{T} \hookrightarrow_C \ id \ \hookrightarrow_C \cong_T .$$

Classifiable non-shallow

Lemma (M. 2023)

Suppose $\kappa = \lambda^+ = 2^\lambda$. *The following reduction is strict. Let* $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. *If* T_1 *is a classifiable non-shallow theory and* T_2 *is a non-classifiable theory, then*

$$\cong^{\lambda}_{T_2} \hookrightarrow_C \cong_{T_1} \hookrightarrow_C \cong_{T_2} .$$

Classifiable shallow

Lemma (M. 2023)

Suppose $\kappa = \lambda^+ = 2^\lambda$. The following reductions are strict. Let $\kappa = \aleph_\gamma$ be such that $\beth_{\omega_1}(|\gamma|) \leq \kappa$. Suppose T_1 is a classifiable shallow theory, T_2 a classifiable non-shallow theory, and T_3 non-classifiable theory. Then

$$\cong_{T_1} \hookrightarrow_B \cong^{\lambda}_{T_3} \hookrightarrow_C \cong_{T_2} .$$

Thank you

Article at: https://arxiv.org/abs/2308.07510

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