Filter Reflection and the Borel reducibility

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Capturing clubs

Suppose X and S are stationary subsets of κ, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S\rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α.

We say that $\vec{\mathcal{F}}$ *captures clubs* iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is non-stationary.

Example. When $\vec{\mathcal{F}}$ is the sequence of club filters, $\vec{\mathcal{F}}$ captures clubs.

Filter reflection

We say that X *$\vec{\mathcal{F}}$-reflects to S* iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ is stationary.

We say that X *$\mathfrak{f}$-reflects to S* iff there exists a sequence of filters $\vec{\mathcal{F}}$ over a stationary subset S' of S such that X $\vec{\mathcal{F}}$-reflects to S'.

Reflection with diamond

We say that X *$\vec{\mathcal{F}}$-reflects with $\diamondsuit$ to S* iff $\vec{\mathcal{F}}$ captures clubs and there exists a sequence $\langle Y_\alpha \mid \alpha \in S\rangle$ such that, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y_\alpha = Y \cap \alpha \;\&\; Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ is stationary.

We say that X *$\mathfrak{f}$-reflects with $\diamondsuit$ to S* whenever there exists a stationary $S' \subseteq S$ and sequence of filters $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S'\rangle$ such that X $\vec{\mathcal{F}}$-reflects with $\diamondsuit$ to S'.

Examples

Vanilla stationary reflection is a particular case of filter reflection, this is when $\vec{\mathcal{F}}$ is the sequence of club filters.

Suppose $V = L$. Then, for every stationary $S \subseteq \kappa$, κ $\mathfrak{f}$-reflects with $\diamondsuit$ to S. In particular, filter reflection holds at any non weakly compact cardinal.

Forcing filter reflection

For all stationary subsets X and S of κ, there exists a $<\kappa$-closed κ^+-cc forcing extension, in which X $\mathfrak{f}$-reflects to S.

Failure of filter reflection

There exists a cofinality-preserving forcing extension in which for all two disjoint stationary subsets X, S of κ, X does not $\mathfrak{f}$-reflect to S.

Forcing-preserving filter reflection

Definition. We say that a sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S\rangle$ is *θ-complete* if the set $\{\alpha \in S \mid \mathcal{F}_\alpha \textit{ is not } \theta\textit{-complete}\}$ is non-stationary.

Theorem. Suppose $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S\rangle$ is a θ-complete sequence. Suppose $\mathbb{P}$ is a forcing notion with θ-cc and κ-stationary-cc, and $X \subseteq \kappa$ is a stationary set such that X $\vec{\mathcal{F}}$-reflects to S. Then $\mathbb{P}$ forces that X $\mathfrak{f}$-reflects to S.

Notice that for any regular uncountable cardinal $\theta < \kappa$, if $S \subseteq cof(\geq\theta)$, then for every $\alpha \in S$, the club filter over α is θ-complete. **Therefore, forcing with θ-cc and κ-stationary-cc cannot destroy vanilla stationary reflection.**

Reducibility

Definition. For $i < 2$, let X_i be some space from the collection $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$.
Let R_0 and R_1 be binary relations over X_0 and X_1, respectively. A function $f : X_0 \to X_1$ is said to be a *reduction of R_0 to R_1* iff, for all $\eta, \xi \in X_0$,

$$\eta\ R_0\ \xi \text{ iff } f(\eta)\ R_1\ f(\xi).$$

The existence of a continuous function f that is a reduction is denoted by $R_0 \hookrightarrow_c R_1$. We likewise define $R_0 \hookrightarrow_B R_1$ once we replace continuous by Borel

Relations

Defintion. For every $\theta \in [2, \kappa]$, the equivalence relation $=^\theta_S$ over θ^κ is defined via

$$\eta =^\theta_S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\} \text{ is non-stationary.}$$

Defintion. Let $\mathcal{F}$ be a filter over α. For every $\theta \in [2, \kappa]$, the equivalence modulo $\mathcal{F}$, $\sim^\theta_{\mathcal{F}}$, over θ^α, is define via

$$(\eta, \xi) \in \sim^\theta_{\mathcal{F}} \text{ iff } \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\} \in \mathcal{F}$$

Theorem 1

If X $\mathfrak{f}$-reflects with $\diamondsuit$ to S, then $=^\kappa_X \hookrightarrow_c =^2_S$.

Theorem 2

Suppose Martin's Maximum (MM) holds, $\kappa \geq \aleph_2$, $X \subseteq \kappa \cap cof(\omega)$ is stationary, and $S = \kappa \cap cof(\omega_1)$. If $\diamondsuit_X$ holds, then X reflects with $\diamondsuit$ to S.

Remark. MM implies that for every successor κ of a singular strong limit of uncountable cofinality, $=^\kappa_X \hookrightarrow_c =^2_S$ for $X = E^\kappa_\omega$ and $S = E^\kappa_{\omega_1}$.

Filtered functions

Definition. For every $\theta, \gamma \in [2, \kappa]$, $F : \theta^\kappa \to \gamma^\kappa$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S\rangle$ a sequence of filters. We say that F captures $\vec{\mathcal{F}}$ if the following holds:
For all $\alpha \in S$ and $\eta, \xi \in \theta^\kappa$, $\eta \restriction \alpha \sim^\theta_{\mathcal{F}_\alpha} \xi \restriction \alpha$ iff $F(\eta)(\alpha) = F(\xi)(\alpha)$.

Main Theorem

Theorem. Let $X, S \subseteq \kappa$ be stationary sets. The following are equivalent:

1. X $\mathfrak{f}$-reflects to S
2. there is $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S\rangle$ and $F : 2^\kappa \to \kappa^\kappa$ a reduction from $=^2_X$ to $=^\kappa_S$ that captures $\vec{\mathcal{F}}$.

Remark. *X stationary reflects to S if and only if there is $F : 2^\kappa \to \kappa^\kappa$ a reduction from $=^2_X$ to $=^\kappa_S$ that captures the sequence of the club filters.*

Question

Is it consistence that $=^\kappa_S \not\hookrightarrow_B =^2_S$ holds for all stationary set $S \subseteq \kappa$?

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