Classification Theory in Generalized Descriptive Set Theory

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Geometry

- Independence of Euclid's fifth postulate, the parallel postulate.
- Khayyám (1077) and Saccheri (1733) considered the three different cases of the Khayyám-Saccheri quadrilateral (right, obtuse, and acute).
- Euclidean geometry, Elliptic geometry, Hyperbolic geometry.

The spectrum fuction

Let T be a countable theory over a countable language. Let $I(T, \alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T, \alpha)$?

Categoricity

- **1904:** Veble introduced categorical theories.
- **1915 - 1920:** Löwenheim-Skolem Theorem.
- **1965:** Morley's categoricity theorem.

Morley's conjecture

1960's: Let T be a first-order countable theory over a countable language. For all $\aleph_0 < \lambda < \kappa$,

$$I(T,\lambda) \leq I(T,\kappa).$$

1990: Shelah proved Morley's conjecture.

Shelah's Main Gap Theorem

Theorem (Shelah 1990)

Either, for every uncountable cardinal α, $I(T,\alpha) = 2^{\alpha}$; or $\forall \alpha > 0$, $I(T,\aleph_\alpha) < \beth_{\omega_1}(|\alpha|)$.

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Descriptive Set Theory

- **1989:** Friedman and Stanley introduced the Borel reducibility between classes of countable structures.
- **1993:** Mekler-Väänänen κ-separation theorem.
- **2014:** Friedman-Hyttinen-Kulikov developed GDST and a systematic comparison between the Main Gap dividing lines and the complexity given by Borel reducibility.

The bounded topology

Let κ be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

The Generalised Baire spaces

The generalised Baire space is the space κ^κ endowed with the bounded topology.

The generalised Cantor space is the subspace 2^κ.

Coding structures

Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$.

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in \kappa^\kappa$ define the structure $\mathcal{A}_f$ with domain κ and for every tuple $(a_1, a_2, \dots, a_n)$ in κ^n

$$(a_1, a_2, \dots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \dots, a_n)) > 0$$

The isomorphism relation

Definition

Given T a first-order complete countable theory in a countable vocabulary, we say that $f, g \in \kappa^\kappa$ are $\cong_T$ equivalent if one of the following holds:

- $\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
- $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *reducible* to E_2, if there is a function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x,y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$. We write $E_1 \hookrightarrow_r E_2$.

We can define a partial order on the set of all first-order complete countable theories

$$T \leq^\kappa T' \text{ iff } \cong_T \hookrightarrow_C \cong_{T'}$$

Non-classifiable theories

A theory T is non-classifiable if it is a countable complete theory that satisfies one of the following:

- T is unstable;
- T is stable unsuperstable;
- T is superstable with DOP;
- T is superstable with OTOP.

First dividing lines

Fact (Friedman-Hyttinen-Kulikov 2014)

1. *Let $\kappa^{<\kappa} = \kappa > 2^\omega$. If T is classifiable and shallow, then $\cong_T$ is κ-Borel.*
2. *If T is classifiable non-shallow, then $\cong_T$ is $\Delta^1_1(\kappa)$ not κ-Borel.*
3. *If T is unstable or stable with the OTOP or superstable with the DOP and $\kappa > \omega_1$, then $\cong_T$ is not $\Delta^1_1(\kappa)$.*
4. *If T is stable unsuperstable, then $\cong_T$ is not κ-Borel.*

Classifiable and shallow

Theorem (Mangraviti - Motto Ros 2020)

Let κ be such that $\kappa > 2^\omega$. If T is classifiable and shallow with depth α, then $rk_B(\cong_T) \leq 4\alpha$.

Theorem (Mangraviti - Motto Ros 2020)

Let $\kappa = \aleph_\gamma$ be such that $\kappa^{<\kappa} = \kappa$ and $\beth_{\omega_1}(|\gamma|) \leq \kappa$. Let T, T' be countable complete first-order theories, and suppose T is classifiable and shallow, while T' is not. Then

$$\cong_T \hookrightarrow_B \cong_{T'}$$

General reduction

Fact (Mangraviti-Motto Ros)

Let E_1 be a Borel equivalence relation with $\gamma \leq \kappa$ equivalence classes and E_2 be an equivalence relation with θ equivalence classes. If $\gamma \leq \theta$, then $E_1 \hookrightarrow_B E_2$.

1_ϱ relation

Let $0 < \varrho \leq \kappa$. $\eta \; 1_\varrho \; \xi$ if and only if one of the following holds:

- ϱ is finite:
  - $\eta(0) = \xi(0) < \varrho - 1$;
  - $\eta(0), \xi(0) \geq \varrho - 1$.
- ϱ is infinite:
  - $\eta(0) = \xi(0) < \varrho$;
  - $\eta(0), \xi(0) \geq \varrho$.

Few equivalence classes

Lemma (M. 2023)

Suppose $\kappa > 2^\omega$ and T is a countable first-order theory in a countable vocabulary (not necessarily complete) such that $\cong_T$ has $\varrho \leq \kappa$ equivalence classes. Then

$$\cong_T \hookrightarrow_B 1_\varrho \text{ and } 1_\varrho \hookrightarrow_L \cong_T .$$

Even more, if T is not categorical then $\cong_T \not\hookrightarrow_C 1_\varrho$.

Gap: Shallow and Non-shallow

Theorem (M. 2023)

Suppose $\aleph_\mu = \kappa = \lambda^+ = 2^\lambda$ *is such that* $\beth_{\omega_1}(|\mu|) \leq \kappa$. *Let* T_1 *be a countable complete classifiable shallow theory with* $\varrho = I(\kappa, T_1)$, T_2 *be a countable complete theory not classifiable shallow. If* T *is classifiable shallow such that* $1 < I(\kappa, T) < I(\kappa, T_1)$, *then*

$$\cong_T \hookrightarrow_B 1_\varrho \hookrightarrow_L \cong_{T_1} \hookrightarrow_B 1_\kappa \hookrightarrow_L \cong_{T_2} .$$

In particular

$$\cong_{T_2} \not\hookrightarrow_r 1_\kappa \not\hookrightarrow_r \cong_{T_1} \not\hookrightarrow_C 1_\varrho \not\hookrightarrow_r \cong_T .$$

Unsuperstable theories

Theorem (Hyttinen - Kulikov - M. 2017)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, and $\lambda^\omega = \lambda$. If T is classifiable and T' is stable unsuperstable, then $T \leq^\kappa T'$ and $T' \not\leq^\kappa T$.

Theorem (M. 2022)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^\omega = \lambda$. If T is a classifiable theory, and T' is an unsuperstable theory, then $T \leq^\kappa T'$ and $T' \not\leq^\kappa T$.

Equivalence modulo γ cofinality

Definition

We define the equivalence relation $=^2_\gamma \subseteq 2^\kappa \times 2^\kappa$, as follows: let $S = \{\alpha < \kappa \mid cf(\alpha) = \gamma\}$,

$$\eta =^2_\gamma \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

Borel-reducibility Main Gap

Theorem (M. 2023)

Let $\mathfrak{c} = 2^\omega$. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{\omega_1}$. If T is a classifiable theory, and T' is a non-classifiable theory, then there is $\gamma < \kappa$ such that

$$\cong_T \hookrightarrow_C =^2_\gamma \hookrightarrow_C \cong_{T'} \text{ and } =^2_\gamma \not\hookrightarrow_B \cong_T .$$

In particular

$$T \ \leq^\kappa \ T' \text{ and } T' \ \not\leq^\kappa \ T.$$

Gap: Classifiable vs Unstable or Superstable

Theorem (M. 2023)

If λ is such that $\lambda = \lambda^{<\lambda}$. For all $\omega < \gamma < \lambda$ regular, T_2 unstable or superstable, and T_1 is classifiable:

$$\cong_{T_1} \hookrightarrow_C =^2_\gamma \hookrightarrow_C \cong_{T_2} \text{ and } \cong_{T_2} \not\hookrightarrow_B \cong_{T_1}.$$

In particular

$$=^2_\gamma \not\hookrightarrow_B \cong_{T_1}.$$

A bigger Gap

Theorem (M. 2023)

Suppose κ is inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{\omega_1}$. There exists a cofinality-preserving forcing extension in which the following holds:
If T_1 is classifiable and T_2 is not. Then there is a regular cardinal $\gamma < \kappa$ such that, if $X, Y \subseteq S^\kappa_\gamma$ are stationary and disjoint, then $=^2_X$ and $=^2_Y$ are strictly in between $\cong_{T_1}$ and $\cong_{T_2}$.

Main Gap Dichotomy

Theorem (M. 2023)

Let κ be inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. There exists a $<\kappa$-closed κ^+-cc forcing extension in which for any countable first-order theory in a countable vocabulary (not necessarily complete), T, one of the following holds:

- $\cong_T$ *is* $\Delta^1_1(\kappa)$*;*
- $\cong_T$ *is* $\Sigma^1_1(\kappa)$*-complete.*

Non-classifiable theories

Lemma (M. 2023)

Let κ be strongly inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. For all cardinals $\aleph_0 < \mu < \delta < \kappa$, if T is a non-classifiable theory then

$$\cong^\mu_T \hookrightarrow_C \cong^\delta_T \hookrightarrow_C id \hookrightarrow_C \cong_T .$$

Classifiable non-shallow

Lemma (M. 2023)

Suppose $\kappa = \lambda^+ = 2^\lambda$. *The following reduction is strict. Let* $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. *If* T_1 *is a classifiable non-shallow theory and* T_2 *is a non-classifiable theory, then*

$$\cong^{\lambda}_{T_2} \hookrightarrow_C \cong_{T_1} \hookrightarrow_C \cong_{T_2} .$$

Classifiable shallow

Lemma (M. 2023)

Suppose $\kappa = \lambda^+ = 2^\lambda$. The following reductions are strict. Let $\kappa = \aleph_\gamma$ be such that $\beth_{\omega_1}(|\gamma|) \leq \kappa$. Suppose T_1 is a classifiable shallow theory, T_2 a classifiable non-shallow theory, and T_3 non-classifiable theory. Then

$$\cong_{T_1} \hookrightarrow_B \cong^\lambda_{T_3} \hookrightarrow_C \cong_{T_2}.$$

Thank you

Article at: https://arxiv.org/abs/2308.07510

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