Shelah's Main Gap Theorem in the Borel-reducibility hierarchy

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Outline

1 Classifying First-order countable Theories

2 Shelah's Main Gap Theorem

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1. Classifying First-order countable Theories
2. Shelah's Main Gap Theorem

The spectrum problem

Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T,\alpha)$?

- **Löwenheim-Skolem Theorem:** $\exists\alpha \geq \omega\ I(T,\alpha) \neq 0 \Rightarrow \forall\beta \geq \omega\ I(T,\beta) \neq 0$.
- **Morley's categoricity:** $\exists\alpha > \omega\ I(T,\alpha) = 1 \Rightarrow \forall\beta > \omega\ I(T,\beta) = 1$
- **Shelah's Main Gap Theorem:** Either, for every uncountable cardinal α, $I(T,\alpha) = 2^{\alpha}$, or $\forall\alpha > 0\ I(T,\aleph_\alpha) < \beth_{\omega_1}(\mid \alpha \mid)$.

Approaches

- Shelah's stability theory.
  Classify the models of T by cardinal invariants and clearly differenciate clearly between the theories that can be classified and those that cannot.

- Descriptive set theory:
  It uses Borel-reducibility and the isomorphism relation to define a partial order on the set of all first-order complete countable theories.

The topology

κ is a cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set 2^κ with the bounded topology. For every $\zeta \in 2^{<\kappa}$, the set

$$[\zeta] = \{\eta \in 2^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

Reductions

A function $f\colon 2^\kappa \to 2^\kappa$ is *Borel*, if for every open set $A \subseteq 2^\kappa$ the inverse image $f^{-1}[A]$ is a Borel subset of 2^κ.

Let E_1 and E_2 be equivalence relations on 2^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f\colon 2^\kappa \to 2^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

We write $E_1 \leqslant_B E_2$.

Coding structures

Fix a language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $\eta \in 2^\kappa$ define the structure $\mathcal{A}_\eta$ with domain κ and for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \Leftrightarrow \eta(\pi(m, a_1, a_2, \ldots, a_n)) = 1$$

Definition (The isomorphism relation)

Given T a first-order complete countable theory in a countable vocabulary, we say that $\eta, \xi \in 2^\kappa$ are $\cong_T^\kappa$ equivalent if

- $\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi$
  or
- $\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T$

Borel-reducibility hierarchy

We can define a partial order on the set of all first-order complete countable theories

$$T \leqslant^{\kappa} T' \textit{ iff } \cong^{\kappa}_{T} \leqslant_{B} \cong^{\kappa}_{T'}$$

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Countable

$T = Th(\mathbb{Q}, \leqslant)$.
T', the theory of vector space over the field of rational numbers.

By the Borel-reducibility hierarchy:

$$T \leqslant^\omega T'$$

$$T' \not\leqslant^\omega T$$

By the stability theory T' is simpler than T.

Shelah's Main Gap Theorem

Theorem (Shelah)

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Uncountable

Under some cardinality assumptions on κ have been proved the following.

Theorem (Friedman, Hyttinen, Kulikov)

If T is unstable and T' is classifiable, then $T \not\leqslant^\kappa T'$.

Theorem

If T is stable unsuperstable and T' is classifiable, then

$$T' \leqslant^\kappa T$$

$$T \not\leqslant^\kappa T'$$

Consistency

Theorem

Let $H(\kappa)$ be the following property: If T is classifiable and T' is not, then $T \leqslant^\kappa T'$ and $T' \not\leqslant^\kappa T$.
The following statements hold:

1. *If $V = L$, then $H(\kappa)$ holds.*
2. *There is a κ-closed forcing notion $\mathbb{P}$ with the κ^+-c.c. which forces $H(\kappa)$.*

Borel-reducibility Counterpart

Theorem

The following statement is consistent:
If T_1 is classifiable and T_2 is not classifiable, then $T_1 \leqslant^\kappa T_2$ and there are 2^κ equivalence relations strictly between $\cong^\kappa_{T_1}$ and $\cong^\kappa_{T_2}$.

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