The Borel reducibility Main Gap

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Two notions

- Model theory notion. Classification theory (Shelah 1990)
- Set theory notion. Borel reducibility (Friedman and Stanley 1989)

Theories

- Classifiable theories are divided into:
  - shallow,
    $$I(T, \aleph_\alpha) < \beth_{\omega_1}(| \alpha |);$$
  - non-shallow,
    $$I(T, \alpha) = 2^\alpha.$$
- Non-classifiable theories

Continuous reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *continuous reducible* to E_2, if there is a continuous function $f:\kappa^\kappa\to\kappa^\kappa$ that satisfies $(x,y)\in E_1 \Leftrightarrow (f(x),f(y))\in E_2$. We write $E_1\hookrightarrow_C E_2$.

We can define a partial order on the set of all first-order complete countable theories

$$T\leq^\kappa T' \textit{ iff } \cong_T\hookrightarrow_C\cong_{T'}$$

Question

Question: What can we say about the reducibility between those dividing lines?

Friedman-Hyttinen-Kulikov
Conjecture: If T is classifiable and T' is non-classifiable, then $T \leq^\kappa T'$ $(\cong_T \hookrightarrow_C \cong_{T'})$.

Borel-reducibility Main Gap

Theorem (M.)

Let $\mathfrak{c} = 2^\omega$*. Suppose* $\kappa = \lambda^+ = 2^\lambda$ *and* $2^\mathfrak{c} \leq \lambda = \lambda^{\omega_1}$*. If* T *is a classifiable theory, and* T' *is a non-classifiable theory, then*

$$T \ \leq^\kappa \ T' \text{ and } T' \ \not\leq^\kappa \ T.$$

Main Gap Dichotomy

Theorem (M.)

Let κ be inaccessible, or $\kappa=\lambda^+=2^\lambda$ and $2^{\mathfrak{c}}\leq\lambda=\lambda^{<\omega_1}$. There exists a $<\kappa$-closed κ^+-cc forcing extension in which for any countable first-order theory in a countable vocabulary (not necessarily complete), T, one of the following holds:

- $\cong_T$ *is* $\Delta_1^1(\kappa)$;
- $\cong_T$ *is* $\Sigma_1^1(\kappa)$*-complete.*

In between

Lemma (M.)

Suppose $\kappa = \lambda^+ = 2^\lambda$. *Let* $\kappa = \aleph_\gamma$ *be such that* $\beth_{\omega_1}(|\gamma|) \leq \kappa$ *and* $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. *Suppose* T_1 *is a classifiable shallow theory,* T_2 *a classifiable non-shallow theory, and* T_3 *non-classifiable theory. Then*

$$\cong_{T_1} \hookrightarrow_B \cong^\lambda_{T_3} \hookrightarrow_C \cong_{T_2} \hookrightarrow_C \cong_{T_3} .$$

Equivalence modulo γ cofinality

Definition

We define the equivalence relation $=^2_\gamma \subseteq 2^\kappa \times 2^\kappa$, as follows: let $S = \{\alpha < \kappa \mid cf(\alpha) = \gamma\}$,

$$\eta =^2_\gamma \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

Classifiable theories

Theorem (Hyttinen - Kulikov - M. 2017)

Assume T is a classifiable theory and let $S = \{\alpha < \kappa \mid cf(\alpha) = \gamma\}$. If $\diamondsuit_S$ holds, then $\cong_T \hookrightarrow_C =^2_\gamma$.

The idea

- Construct the reductions $(=^2_\gamma \hookrightarrow_C \cong_T)$.
- Construct Ehrenfeucht-Mostowski models, such that

$$f \ =^2_\gamma \ g \text{ iff } \mathcal{M}^f \cong \mathcal{M}^g.$$

- Construct ordered trees, such that

$$f \ =^2_\gamma \ g \ \Leftrightarrow A_f \cong A_g \Leftrightarrow \mathcal{M}^f \cong \mathcal{M}^g.$$

Ordered trees from the linear order

- ω_1-dense,
- (κ, ω_1)-nice, $(< \kappa)$-stable,
- κ-colorable.

The F^{φ}_{ω} isolation

Definition

Let $\varphi(x, y) :=$ "$y > x$", we define F^{φ}_{ω} in the following way. Let $|B| < \kappa$ and $p \in S_{bs}(B)$, $(p, A) \in F^{\varphi}_{\omega}$ if and only if $A \subseteq B$, A is finite, and there is $a \in A$ such that

$$\{a > x, x = a\} \cap p \neq \emptyset \ \& \ a \models p \restriction B \backslash \{a\}.$$

F^{φ}_{ω} saturation

Definition

C is $(F^{\varphi}_{\omega}, \kappa)$-saturated if for all $B \subseteq C$ of size smaller than κ, and $p \in S_{bs}(B)$, $(p, A) \in F^{\varphi}_{\omega}$ implies that p is realized in C.

$(F^{\varphi}_{\omega}, \kappa)$-saturated model

Lemma (M.)

Let $\mathfrak{c} = 2^{\omega}$. Suppose $\kappa = \lambda^{+} = 2^{\lambda}$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{\omega_1}$. There is an $(F^{\varphi}_{\omega}, \kappa)$-saturated model over $\mathcal{I}^0$ and it is an ω_1-dense, (κ, ω_1)-nice, $(< \kappa)$-stable, and κ-colorable linear order.

Thus

$$\cong_T \hookrightarrow_C =^2_{\gamma} \hookrightarrow_C \cong_{T'} .$$

Thank you

Article at: https://arxiv.org/abs/2308.07510