Ordered trees and the kappa-Borel reducibility of unsuperstable theories

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Shelah's Main Gap Theorem

Let $I(T, \alpha)$ denote the number of non-isomorphic models of T with cardinality α.

Theorem (Shelah)

Either, for every uncountable cardinal α, $I(T, \alpha) = 2^{\alpha}$, or $\forall \alpha > 0\ I(T, \aleph_\alpha) < \beth_{\omega_1}(|\alpha|)$.

Theorem (Shelah)

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

The topology

κ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set 2^κ with the bounded topology. For every $\zeta \in 2^{<\kappa}$, the set

$$[\zeta] = \{\eta \in 2^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

Coding structures

Fix a language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in 2^\kappa$ define the structure $\mathcal{A}_f$ with domain κ and for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

Definition (The isomorphism relation)

Given T a first-order complete countable theory in a countable vocabulary, we say that $f, g \in 2^\kappa$ are $\cong^\kappa_T$ equivalent if
$\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
or $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

Reductions

Let E_1 and E_2 be equivalence relations on 2^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f : 2^\kappa \to 2^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.
We write $E_1 \hookrightarrow_b^\kappa E_2$.

We can define a partial order on the set of all first-order complete countable theories

$$T \leq^\kappa T' \textit{ iff } \cong^\kappa_T \hookrightarrow^\kappa_b \cong^\kappa_{T'}$$

The question

Question:
Is there a Borel reducibility counterpart of the Main Gap Theorem in the generalized descriptive set theoretical approach?

non-classifiable theories

A theory T is non-classifiable if it is a countable complete theory that satisfies one of the following:

- T is unstable;
- T is stable unsuperstable;
- T is superstable with DOP;
- T is superstable with OTOP.

Progress

Theorem (Mangraviti - Motto Ros)

Let $\kappa = \aleph_\gamma$ be such that $\kappa^{<\kappa} = \kappa$ and $\beth_{\omega_1}(|\gamma|) \leq \kappa$. Let T, T' be countable complete first-order theories, and suppose T is classifiable and shallow, while T' is not. Then

$$T \ \leq^\kappa \ T'$$

Theorem (Friedman - Hyttinen - Kulikov)

If T is classifiable and T' is unsuperstable, then

$$T' \not\leq^\kappa T$$

Progress

Theorem (Hyttinen - Kulikov - Moreno)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, *and* $\lambda^{<\lambda} = \lambda$. *There is a* κ*-closed* κ^+*-cc forcing which forces:*
If T *is classifiable and* T' *is not, then* $T \leq^\kappa T'$ *and* $T' \not\leq^\kappa T$

Theorem (Fernandes - Moreno - Rinot)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, *and* $\lambda^{<\lambda} = \lambda$. *Let* T *be a non-classifiable theory. There is a* κ*-closed* κ^+*-cc forcing which forces:*
If T' *is a countable complete first-order theory, then* $T' \leq^\kappa T$.

Stable unsuperstable theories

Theorem (Hyttinen - Kulikov - Moreno)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, and $\lambda^{<\lambda} = \lambda$. If T is classifiable and T' is stable unsuperstable, then $T \leq^\kappa T'$.

Equivalence modulo ω cofinality

Definition

We define the equivalence relation $=^2_\omega \subseteq 2^\kappa \times 2^\kappa$, as follows: let $S = \{\alpha < \kappa \mid cf(\alpha) = \omega\}$,

$$\eta =^2_\omega \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

Classifiable theories

Theorem (Hyttinen - Kulikov - Moreno)

Assume T is a countable complete classifiable theory over a countable vocabulary. Suppose $\kappa=\lambda^+$, $2^\lambda>2^\omega$, and $\lambda^{<\lambda}=\lambda$. Then $\cong^\kappa_T\hookrightarrow^\kappa_b=^2_\omega$.

Coloured trees

Definition

Let β an ordinal smaller or equal to κ. A coloured tree is a pair (t, c), where t is a κ^+, $(\omega + 2)$-tree and c is a map $c : leaves(t) \to \beta$ (the color function).

Theorem (Hyttinen - Kulikov)

For all $f \in 2^\kappa$ there is a coloured tree J_f with two colors, such that for all $f, g \in 2^\kappa$ the following holds:

$$f \ =^2_\omega \ g \ \Leftrightarrow J_f \cong J_g.$$

Ordered trees

Definition

Let K^{ω}_{tr} be the class of models $(A, \prec, (P_n)_{n\leq\omega}, <)$, where:

- there is a linear order $(I, <_I)$ such that $A \subseteq I^{\leq\omega}$;
- $(A, \prec)$ is a tree with unique limits;
- let $lg(\eta)$ be the length of η (i.e. the domain of η) and $P_n = \{\eta \in A \mid lg(\eta) = n\}$ for $n \leq \omega$;

Ordered trees

Definition (continuation)

Let K^{ω}_{tr} be the class of models $(A, \prec, (P_n)_{n\leq\omega}, <)$, where:

- for every $\eta \in A$ with $lg(\eta) < \omega$, define $Suc_A(\eta)$ as $\{\xi \in A \mid \eta \prec \xi \wedge lg(\xi) = lg(\eta) + 1\}$. If $\xi < \zeta$, then there is $\eta \in A$ such that $\xi, \zeta \in Suc_A(\eta)$;
- for every $\eta \in A \backslash P_\omega$, $<\restriction Suc_A(\eta)$ is the induced linear order from I, i.e.

$$\eta^\frown\langle x\rangle < \eta^\frown\langle y\rangle \Leftrightarrow x <_I y;$$

Coloring orders

Definition

Let I be a linear order of size κ. We say that I is κ-colorable if there is a function $F : I \to \kappa$ such that for all $B \subseteq I$, $|B| < \kappa$, $b \in I \backslash B$, and $p = tp_{bs}(b, B, I)$ such that the following hold: For all $\alpha \in \kappa$, $|\{a \in I \mid a \models p \;\&\; F(a) = \alpha\}| = \kappa$.

Theorem

Suppose I is a κ-colorable linear order. Then for any $f \in 2^\kappa$, there is an ordered tree $A_f(I)$ that satisfies:
For all $f, g \in 2^\kappa$,

$$f \ =^2_\omega \ g \ \Leftrightarrow A_f(I) \cong A_g(I).$$

The isomorphism

Theorem (Shelah)

Suppose T is a countable complete unsuperstable theory in a countable vocabulary.
If κ is a regular uncountable cardinal, $A_1, A_2 \in K^{\omega}_{tr}$ have size κ, A_1, A_2 are locally (κ, bs, bs)-nice and $(<\kappa, bs)$-stable, $EM(A_1, \Phi)$ is isomorphic to $EM(A_2, \Phi)$, then $S(A_1) =^2_\omega S(A_2)$.

In our construction, $S(A_f(I)) =^2_\omega S(A_g(I))$ is equivalent $f =^2_\omega g$.

Question

Question: Is there a κ-colorable linear order I such that for all $f \in 2^\kappa$, $A_f(I)$ is locally (κ, bs, bs)-nice and $(< \kappa, bs)$-stable?

κ-representation

Definition

Let A be an arbitrary set of size at most κ. A sequence $\mathbb{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ is a κ-representation of A, if $\langle A_\alpha \mid \alpha < \kappa \rangle$ is an increasing continuous sequence of subsets of A, for all $\alpha < \kappa$, $|A_\alpha| < \kappa$, and $\bigcup_{\alpha<\kappa} A_\alpha = A$.

Nice linear order

Definition (Lemma by Hyttinen - Tuuri)

Let I be a linear order of size κ and $\langle I_\alpha \mid \alpha < \kappa \rangle$ a κ-representation. If there is a club $C \subseteq \kappa$, such that for all limit $\delta \in C$, for all $x \in I$ there is $\beta < \delta$ such that:

$$\forall \sigma \in I_\delta [\sigma \geq x \Rightarrow \exists \sigma' \in I_\beta \ (\sigma \geq \sigma' \geq x)]$$

Then I is (κ, bs, bs)-nice and for all $f \in 2^\kappa$, $A_f(I)$ is locally (κ, bs, bs)-nice.

Locally nice ordered tree

Definition

$A \in K^{\omega}_{tr}$ of size at most κ, is locally (κ, bs, bs)-nice if for every $\eta \in A \backslash P^A_{\omega}$, $(Suc_A(\eta), <)$ is (κ, bs, bs)-nice, $Suc_A(\eta)$ is infinite, and there is $\xi \in P^A_{\omega}$ such that $\eta \prec \xi$.

Stable ordered tree

Definition

$A \in K^{\omega}_{tr}$ is $(< \kappa, bs)$-stable if for every $B \subseteq A$ of size smaller than κ,

$$\kappa > |\{tp_{bs}(a, B, A) \mid a \in A\}|.$$

The order

Theorem

There is a $(<\kappa, bs)$-stable (κ, bs, bs)-nice κ-colorable linear order.

Construction

Definition

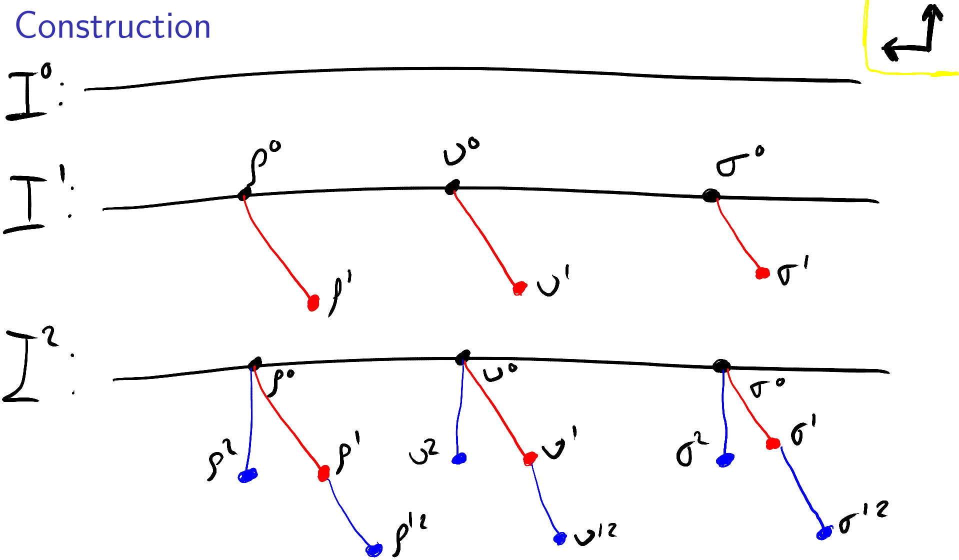
Let $\mathbb{Q}$ be the linear order of the rational numbers.
Let $\kappa \times \mathbb{Q}$ be order by the lexicographic order, I^0 be the set of functions $f : \omega \to \kappa \times \mathbb{Q}$ such that $f(n) = (f_1(n), f_2(n))$, for which $\{n \in \omega \mid f_1(n) \neq 0\}$ is finite.
If $f, g \in I^0$, then $f < g$ if and only if $f(n) < g(n)$, where n is the least number such that $f(n) \neq g(n)$.

I^0

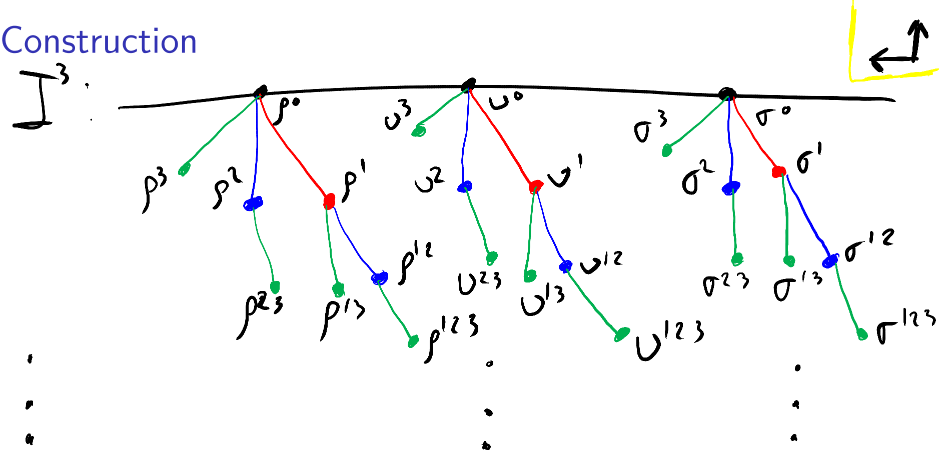
Lemma

I^0 is a $(<\kappa, bs)$-stable (κ, bs, bs)-nice linear order.

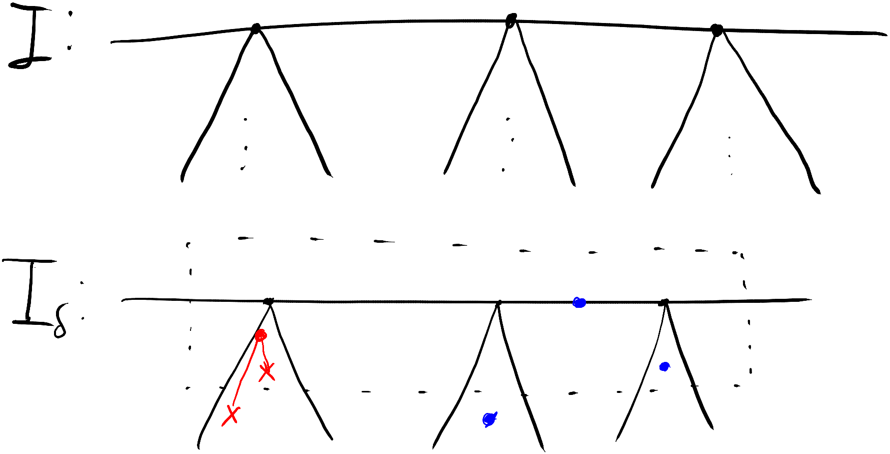
Construction



Construction



Construction



Corollary

Theorem

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^\omega = \lambda$. If T_1 is a countable complete classifiable theory, and T_2 is a countable complete unsuperstable theory, then $T_1 \leq^\kappa T_2$.

Theorem

There exists a $< \kappa$-closed κ^+-cc forcing extension in which for all countable complete unsuperstable theory T, $\cong^\kappa_T$ is Σ^1_1-complete.

The paper entitled **On unsuperstable theories in GDST** can be found at:

https://arxiv.org/abs/2203.14292

Thank you