Diamond sharp and the Generalized Baire Spaces

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Our paper, entitled **Inclusion modulo nonstationary** is available at `https://arxiv.org/abs/1906.10066`

A month ago, the name of our preprint was *Analytic quasi-orders and two forms of diamond* since we had one diamond principle for non-ineffable sets and one for ineffable sets. In the meantime, we found a single principle that works uniformly.

Outline

A Quasi-order

A quasi-order is a binary relation which is reflexive and transitive.

Definition

The quasi-order $\leqslant^*$ over the Baire space $\mathbb{N}^{\mathbb{N}}$ is defined as follows:

$$\eta \leqslant^* \xi \Leftrightarrow \{n \in \mathbb{N} \mid \eta(n) > \xi(n)\} \text{ is finite.}$$

Theorem (Hechler, 1974)

The structure $(\mathbb{N}^{\mathbb{N}}, \leqslant^*)$ is universal in the following sense:
For any σ-directed poset $\mathbb{P}$ with no maximal element, there is a ccc forcing extension in which $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ contains a cofinal order-isomorphic copy of $\mathbb{P}$.

A Refinement of $\leqslant^*$

Definition

Given a stationary subset $S \subseteq \kappa$, we define a quasi-order $\leqslant^S$ over κ^κ by letting, for any two elements $\eta : \kappa \to \kappa$ and $\xi : \kappa \to \kappa$,

$$\eta \leqslant^S \xi \textit{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \textit{ is nonstationary.}$$

Galvin and Hajnal (1975) used this order to attach a rank $\|\eta\|$ to each η, in studying the behavior of the power function over the singular cardinals.

Comparing $\leq^S$

How $\leq^S$ compares with $\leq^{S'}$, for different S and S'?

Theorem

Assume that κ is a regular uncountable cardinal and GCH *holds. Then there exists a cofinality-preserving* GCH*-preserving forcing extension in which for all stationary subsets S, S' of κ, there exists a map $f : \kappa^{\leq\kappa} \to \kappa^{\leq\kappa}$ such that, for all $\eta, \xi \in \kappa^{\leq\kappa}$,*

- $\mathrm{dom}(f(\eta)) = \mathrm{dom}(\eta)$;
- *if $\eta \subseteq \xi$, then $f(\eta) \subseteq f(\xi)$;*
- *if $\mathrm{dom}(\eta) = \mathrm{dom}(\xi) = \kappa$, then $\eta \leq^S \xi$ iff $f(\eta) \leq^{S'} f(\xi)$.*

Outline

The Generalized Baire Space

The generalized Baire space is the set κ^κ endowed with the bounded topology, in which a basic open set takes the form $[\zeta] := \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$, with ζ, an element of $\kappa^{<\kappa}$.

A subset $F \subseteq \kappa^\kappa$ is *closed* iff its complement is open iff there exists a tree $T \subseteq \kappa^{<\kappa}$ such that $[T] := \{\eta \in \kappa^\kappa \mid \forall\alpha < \kappa(\eta \restriction \alpha \in T)\}$ is equal to F.

A subset $A \subseteq \kappa^\kappa$ is *analytic* iff there is a closed subset F of the product space $\kappa^\kappa \times \kappa^\kappa$ such that its projection $\mathrm{pr}(F) := \{\eta \in \kappa^\kappa \mid \exists\xi \in \kappa^\kappa \ (\eta, \xi) \in F\}$ is equal to A.

The Generalized Cantor Space

The generalized Cantor space is the subspace 2^κ of κ^κ endowed with the induced topology.

The notions of open, closed and analytic subsets of 2^κ, $2^\kappa \times 2^\kappa$ and $\kappa^\kappa \times \kappa^\kappa$ are then defined in the obvious way.

Definition

The restriction of the quasi-order $\leq^S$ to 2^κ is denoted by $\subseteq^S$.

Lipschitz Reduction

Let Q_1 and Q_2 be quasi-orders on $X, Y \in \{2^\kappa, \kappa^\kappa\}$ respectively.
We say that Q_1 is 1-*Lipschitz reducible* to Q_2 iff there is a function $f\colon X \to Y$ that satisfies for all $a, b \in X$:

- $(a, b) \in Q_1 \iff (f(a), f(b)) \in Q_2$;
- $\forall \alpha \leq \kappa \; (a \restriction \alpha = b \restriction \alpha) \implies (f(a) \restriction \alpha = f(b) \restriction \alpha)$.

We write $Q_1 \hookrightarrow_1 Q_2$.

The Universality of $\subseteq^S$

Theorem

Assume that κ is a regular uncountable cardinal and GCH *holds. Then there exists a cofinality-preserving* GCH*-preserving forcing extension in which, for every analytic quasi-order Q over κ^κ and every stationary $S \subseteq \kappa$, $Q \hookrightarrow_1 \subseteq^S$.*

The universality statement under consideration is optimal, as $Q \hookrightarrow_1 \subseteq^S$ implies that Q analytic.

Outline

The Universality Implications

Before we define the principle $\mathrm{DI}_S^*(\Pi_2^1)$, let us see the universality implications of it and the history behind the abstract definition of $\mathrm{DI}_S^*(\Pi_2^1)$.

Theorem

Suppose S is a stationary subset of a regular uncountable cardinal κ. If $\mathrm{DI}_S^*(\Pi_2^1)$ holds, then, for every analytic quasi-order Q over κ^κ, $Q \hookrightarrow_1 \subseteq^S$.

Time Line

(1) Friedman, Hyttinen, and Kulikov (2014) identified a reflection principle while working on the question *"Is every analytic set a Borel* set?"*. They found a positive answer under the assumption "$V = L$".

(2) Hyttinen and Kulikov (2015) used this principle to show the universality of the symmetric version $=^S$ of $\leq^S$ for $S = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \omega\}$, assuming "$V = L$".

(3) Hyttinen, Kulikov, and Moreno (2019) merged the principle of (1) with a diamond sequence to answer, under the assumption "$V = L$", the question *"Is it consistently true that $\subseteq^S$ is universal for $S = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \omega\}$"*.

Time Line

(4) At the level of large cardinals, working on the consistency of *"Every Borel* set is analytic and co-analytic"*, Asperó, Hyttinen, Kulikov, and Moreno (2019) proved: If κ is a Π^1_2-indescribable cardinal, and $S = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \alpha\}$, then the symmetric version $=^S$ of $\leq^S$ is universal.

Questions

- Can the assumption $V = L$ exchange for Π^1_2-reflection in Hyttinen-Kulikov-Moreno (2019)?
- How can we merge the diamond principle in Hyttinen-Kulikov-Moreno (2019) with Π^1_2-reflection?

Diamond Sharp

For sets N and x, we say that N *sees* x iff N is transitive, p.r.-closed, and $x \cup \{x\} \subseteq N$

Definition (Devlin, 1982)

Let κ be a regular and uncountable cardinal. $\diamondsuit^{\sharp}_{\kappa}$ asserts the existence of a sequence $\langle N_\alpha \mid \alpha < \kappa \rangle$ such that:

1. *for every infinite $\alpha < \kappa$, N_α is a set of cardinality $|\alpha|$ that sees α;*
2. *for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C$, $X \cap \alpha, C \cap \alpha \in N_\alpha$;*
3. *for every Π^1_2-sentence ϕ valid in a structure $\langle \kappa, \in, (A_n)_{n<\omega} \rangle$, there exists $\alpha < \kappa$, such that*

$$N_\alpha \models \text{``}\phi \text{ is valid in } \langle \alpha, \in, (A_n \restriction \alpha)_{n<\omega} \rangle.\text{''}$$

Good and Bad News

The good news

Devlin proved that $\diamondsuit^\sharp_\kappa$ holds in L for every regular uncountable cardinal κ that is not ineffable.

The bad news

For every ineffable cardinal κ, $\diamondsuit^\sharp_\kappa$ fails.
Even a restricted version $\diamondsuit^\sharp_S$ for $S \subseteq \kappa$ will still fail for any ineffable S.

Conclusion

We need a finer principle.

A finer principle

Definition

Let κ be a regular and uncountable cardinal and $S \subseteq \kappa$ a stationary set. $\mathrm{DI}^*_S(\Pi^1_2)$ *asserts the existence of a sequence $\langle N_\alpha \mid \alpha \in S\rangle$ such that:*

1. *for every $\alpha \in S$, N_α is a set of cardinality $< \kappa$ that sees α;*
2. *for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$;*
3. *for every Π^1_2-sentence ϕ valid in a structure $\langle \kappa, \in, (A_n)_{n<\omega}\rangle$, there exists $\alpha \in S$, such that $|N_\alpha| = |\alpha|$ and*

$$N_\alpha \models \text{“}\phi \text{ is valid in } \langle \alpha, \in, (A_n \restriction \alpha)_{n<\omega}\rangle.\text{”}$$

Always on the good side

In L, $\mathrm{DI}^*_S(\Pi^1_2)$ holds for every $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and every stationary $S \subseteq \kappa$.

Local Club Condensation

The Local Club Condensation (LCC) principle was defined and used by Friedman and Holy (2011) to study comparability of large cardinals with inner-type models. LCC provides us with tools to study $\mathrm{DI}^*_S(\Pi^1_2)$. Friedman and Holy proved that the LCC can be obtained everywhere by a class forcing. A set-forcing was then devised by Holy, Welch and Wu:

Theorem (Holy-Welch-Wu, 2015)

Assume GCH. *For every regular cardinal κ, there is a (set-size) notion of forcing $\mathbb{P}$ which is $(<\kappa)$-directed-closed and has the κ^+−cc such that, in $V^{\mathbb{P}}$, the two holds:*

1. *there is $\vec{M}$ such that $\langle H_{\kappa^+}, \in, \vec{M}\rangle \models \mathrm{LCC}(\kappa, \kappa^+]$, and*
2. *there is a Δ_1-formula Θ and a parameter $a \subseteq \kappa$ such that the order defined by $x <_\Theta y \leftrightarrow H_{\kappa^+} \models \Theta(x, y, a)$ is a global well-order of H_{κ^+}.*

Forcing $\mathrm{DI}^*_S(\Pi^1_2)$

Theorem

Suppose that κ is a regular uncountable cardinal, and $\vec{M}$ is such that $\langle H_{\kappa^+}, \in, \vec{M}\rangle \models \mathrm{LCC}(\kappa, \kappa^+]$. Suppose further that there is a subset $a \subseteq \kappa$ and a formula $\Theta \in \Sigma_\omega$ which defines a well-order $<_\Theta$ in H_{κ^+} via $x <_\Theta y$ iff $H_{\kappa^+} \models \Theta(x, y, a)$. Then, for every stationary $S \subseteq \kappa$, $\mathrm{DI}^*_S(\Pi^1_2)$ holds.

Corollary

Assume GCH. For every regular cardinal κ, there is a (set-size) notion of forcing $\mathbb{P}$ which is $({<}\kappa)$-directed-closed and has the $\kappa^+{-}cc$ such that, in $V^{\mathbb{P}}$, for every stationary $S \subseteq \kappa$, $\mathrm{DI}^*_S(\Pi^1_2)$ holds.

Outline

The Universality

Recall

Suppose S is a stationary subset of a regular uncountable cardinal κ. If $\mathsf{DI}^*_S(\Pi^1_2)$ *holds, then, for every analytic quasi-order Q over κ^κ, $Q \hookrightarrow_1 \subseteq^S$.*

Corollary

Assume that κ is a regular uncountable cardinal and GCH *holds. Then there is a $(<\kappa)$-directed-closed, κ^+-cc notion of forcing $\mathbb{P}$ such that, in $V^\mathbb{P}$,* GCH *holds and for every analytic quasi-order Q over κ^κ and every stationary $S \subseteq \kappa$, $Q \hookrightarrow_1 \subseteq^S$.*

Σ^1_1-completeness

Definition

A quasi-order $\trianglelefteq$ over a space $X \in \{2^\kappa, \kappa^\kappa\}$ is said to be Σ^1_1-complete iff it is analytic and, for every analytic quasi-order Q over X, there exists a κ-Borel function $f : X \to X$ reducing Q to $\trianglelefteq$.

Remark

As Lipschitz $\Longrightarrow$ continuous $\Longrightarrow$ κ-Borel, each $\subseteq^S$ is a Σ^1_1-complete quasi-order. Such a consistency was previously only known for S's of one of two specific forms, and the witnessing maps were not Lipschitz.

More on Universality

By the use of canonical functions coding (Friedman) or Kurepa tree coding (Lücke): For any given quasi-order R over κ^κ, there is a forcing extension in which:

1. R is an analytic quasi-order, and
2. for every analytic quasi-order Q over κ^κ and every stationary $S \subseteq \kappa$, $Q \hookrightarrow_1 \subseteq^S$.

So the main advantage of going outside of L is that we can change the quasi-orders that belong to the class of analytic sets.

Conclusions

Suppose $\mathrm{DI}^*_{\kappa\cap\mathrm{cof}\,\lambda}(\Pi^1_2)$ holds and T is a first-order countable relational theory (not necessarily complete).

- If $\lambda=\aleph_0$, then Borel$^* = \Sigma^1_1$.
- If $\lambda=\aleph_0$, κ is $\aleph_0$-inaccessible, then the embedability of linear orders is Σ^1_1-complete.
- If $\lambda=\aleph_0$, κ is $\aleph_0$-inaccessible, and T is complete stable unsuperstable, then $\cong_T$ is Σ^1_1-complete.
- If $\lambda=2^{\aleph_0}$, κ is inaccessible, and T is complete superstable with S-DOP, then $\cong_T$ is Σ^1_1-complete.
- If $\kappa=\lambda^+$, $\lambda^{<\lambda}=\lambda$, and T is complete unstable or superstable with OTOP, then $\cong_T$ is Σ^1_1-complete.
- If $\kappa=\lambda^+>\aleph_1$, $\lambda^{<\lambda}=\lambda$, and T is complete superstable with DOP, then $\cong_T$ is Σ^1_1-complete.
- If $\kappa=\lambda^+$, $\lambda^{<\lambda}=\lambda>\aleph_0$, and $\mathrm{DI}^*_{\kappa\cap\mathrm{cof}\,\aleph_0}(\Pi^1_2)$, then $\cong_T$ is either Δ^1_1 or Σ^1_1-complete.

We thus feel that we have identified the correct combinatorial principle behind a line of results that were previously obtained under the heavy hypothesis of "$V = L$".

Thank you