$\diamondsuit^{\#}_{\kappa}$ and a model theory dichotomy in GDST

Miguel Moreno
(joint work with Gabriel Fernandes and Assaf Rinot)
Bar-Ilan University

Arctic Set Theory Workshop 2019

January 2019

Outline

Shelah's Main Gap Theorem

Theorem (Main Gap, Shelah)

Let T be a first order complete theory in a countable vocabulary and $I(T,\alpha)$ the number of non-isomorphic models of T with cardinality $|\alpha|$. Either, for every uncountable cardinal α, $I(T,\alpha)=2^\alpha$, or $\forall\alpha>0$ $I(T,\aleph_\alpha)<\beth_{\omega_1}(|\alpha|)$.

Theorem (Shelah)

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Questions

What can we say about the complexity of two different non-classifiable theories?

By non-classifiable theories we mean:

- Unstable theories.
- Stable unsuperstale theories.
- Superstable theories with DOP.
- Superstable theories with OTOP.

Have all the non-classifiable theories the same complexity?

Outline

The approach

Use Borel-reducibility and the isomorphism relation on models of size κ to define a partial order on the set of all first-order complete countable theories.

The Generalized Cantor space

κ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

The generalized Cantor space is the set 2^κ with the bounded topology. For every $\zeta \in 2^{<\kappa}$, the set

$$[\zeta] = \{\eta \in 2^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

κ-Borel sets

The collection of κ-Borel subsets of 2^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ.

A function $f : 2^\kappa \to 2^\kappa$ is κ-*Borel*, if for every open set $A \subseteq 2^\kappa$ the inverse image $f^{-1}[A]$ is a κ-Borel subset of 2^κ.

Borel reduction

Let E_1 and E_2 be equivalence relations on 2^κ. We say that E_1 is *Borel reducible* to E_2, if there is a κ-Borel function $f\colon 2^\kappa \to 2^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

We write $E_1 \leqslant_B E_2$.

Coding structures

Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in 2^\kappa$ define the structure $\mathcal{A}_f$ with domain κ and for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \ldots, a_n)) = 1$$

Definition (The isomorphism relation)

Given T a first-order countable theory in a countable vocabulary, we say that $f, g \in 2^\kappa$ are $\cong_T$ equivalent if

- $\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
  or
- $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

The Borel-reducibility hierarchy

We can define a partial order on the set of all first-order countable theories

$$T \leqslant_\kappa T' \textit{ iff } \cong_T \leqslant_B \cong_{T'}$$

Questions

Is the Borel reducibility notion of complexity a refinement of the complexity notion from stability theory?

- If T is a classifiable theory and T' is not, then $T \leqslant_\kappa T'$?
- If T is an unstable theory and T' is not, then $T' \leqslant_\kappa T$?
- Are all the theories comparable by the Borel reducibility notion of compleity, for every two theories T and T' either $T \leqslant_\kappa T'$ or $T' \leqslant_\kappa T$ holds?

Unstable Theories

Theorem (Friedman, Hyttinen, Kulikov)

If T is unstable and T' is classifiable, then $T \not\leqslant_\kappa T'$.

Theorem (Asperó, Hyttinen, Kulikov, Moreno)

Let DLO be the theory of dense linear order without end points. If κ is a Π^1_2-indescribable cardinal, then $T \leqslant_\kappa DLO$ holds for every theory T.

A Borel reducibility counterpart

Let $H(\kappa)$ be the following property: If T is classifiable and T' is not, then $T \leqslant_\kappa T'$ and $T' \nleqslant_\kappa T$.

Theorem (Hyttinen, Kulikov, Moreno)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$.

1. *If $V = L$, then $H(\kappa)$ holds.*
2. *It can be forced that $H(\kappa)$ holds and there are 2^κ equivalence relations strictly between $\cong_T$ and $\cong_{T'}$.*

Outline

$E^2_{\lambda\text{-club}}$

For every regular cardinal $\lambda < \kappa$, the relation $E^2_{\lambda\text{-club}}$ is defined as follow.

Definition

On the space 2^κ, we say that $f, g \in 2^\kappa$ are $E^2_{\lambda\text{-club}}$ equivalent if the set $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\}$ contains an unbounded set closed under λ-limits.

Non-classifiable theories

Theorem (Friedman, Hyttinen, Kulikov)

Suppose that $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$.

1. *If T is unstable or superstable with OTOP, then $E^2_{\lambda\text{-}club} \leqslant_B \cong_T$.*
2. *If $\lambda \geq 2^\omega$ and T is superstable with DOP, then $E^2_{\lambda\text{-}club} \leqslant_B \cong_T$.*

Theorem (Friedman, Hyttinen, Kulikov)

Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and T is a stable unsuperstable theory. Then $E^2_{\omega\text{-}club} \leqslant_B \cong_T$.

Classifiable theories

Theorem (Hyttinen, Kulikov, Moreno)

Suppose T is a classifiable theory, $\lambda < \kappa$ a regular cardinal such that $\diamondsuit_\kappa(cof(\lambda))$ holds. Then $\cong_T \leqslant_B E^2_{\lambda\text{-}club}$.

Outline

Σ_1^1-completeness

An equivalence relation E on 2^κ is Σ_1^1 or *analytic*, if E is the projection of a closed set in $2^\kappa \times 2^\kappa \times 2^\kappa$ and it is Σ_1^1-*complete* or *analytic complete* if it is Σ_1^1 (analytic) and every Σ_1^1 (analytic) equivalence relation is Borel reducible to it.

Working in L

Definition

- *We define a class function $F_\diamondsuit : On \to L$. For all α, $F_\diamondsuit(\alpha)$ is a pair (X_α, C_α) where $X_\alpha, C_\alpha \subseteq \alpha$, C_α is a club if α is a limit ordinal and $C_\alpha = \emptyset$ otherwise. We let $F_\diamondsuit(\alpha) = (X_\alpha, C_\alpha)$ be the $<_L$-least pair such that for all $\beta \in C_\alpha$, $X_\beta \neq X_\alpha \cap \beta$ if α is a limit ordinal and such pair exists and otherwise we let $F_\diamondsuit(\alpha) = (\emptyset, \emptyset)$.*
- *We let $C_\diamondsuit \subseteq On$ be the class of all limit ordinals α such that for all $\beta < \alpha$, $F_\diamondsuit \restriction \beta \in L_\alpha$. Notice that for every regular cardinal α, $C_\diamondsuit \cap \alpha$ is a club.*

Working in L

Definition

For all regular cardinal α and set $A \subset \alpha$, we define the sequence $(X_\gamma, C_\gamma)_{\gamma \in A}$ as the sequence $(F_\diamondsuit(\gamma))_{\gamma \in A}$, and the sequence $(X_\gamma)_{\gamma \in A}$ as the sequence of sets X_γ such that $F_\diamondsuit(\gamma) = (X_\gamma, C_\gamma)$ for some C_γ.

By ZF^- we mean $ZFC + (V = L)$ without the power set axiom. By $ZF^\diamondsuit$ we mean ZF^- with the following axiom:
"For all regular ordinals $\mu < \alpha$ if $(S_\gamma, D_\gamma)_{\gamma \in \alpha}$ is such that for all $\gamma < \alpha$, $F_\diamondsuit(\gamma) = (S_\gamma, D_\gamma)$, then $(S_\gamma)_{\gamma \in cof(\mu)}$ is a diamond sequence."

The Key Lemma

Lemma (Hyttinen, Kulikov, Moreno)

$(V = L)$ *For any* Σ_1*-formula* $\varphi(\eta, \xi, x)$ *with parameter* $x \in 2^\kappa$*, a regular cardinal* $\mu < \kappa$*, the following are equivalent for all* $\eta, \xi \in 2^\kappa$*:*

- $\varphi(\eta, \xi, x)$
- $S \backslash A$ *is non-stationary, where* $S = \{\alpha \in cof(\mu) \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$ *and*

  $$A = \{\alpha \in C_\diamondsuit \cap \kappa \mid \exists\beta > \alpha(L_\beta \models ZF^\diamond \wedge \varphi(\eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha))\}$$

  where $r(\alpha)$ *is the formula "*α *is a regular cardinal".*

The dichotomy

Theorem (Hyttinen, Kulikov, Moreno)

$(V = L)$ *For every* $\lambda < \kappa$ *regular,* $E^2_{\lambda\text{-club}}$ *is a* Σ^1_1*-complete equivalence relation.*

Theorem (Hyttinen, Kulikov, Moreno)

$(V = L)$ *Suppose that* κ *is the successor of a regular uncountable cardinal. If* T *is a theory in a countable vocabulary. Then one of the following holds.*

- $\cong_T$ *is* Δ^1_1 *(all the complete extensions of* T *are classifiable).*
- $\cong_T$ *is* Σ^1_1*-complete (*T *has at least one non-classifiable extension).*

Notice that T is not required to be complete.

Outline

$\diamondsuit_\kappa^\#(cof(\mu))$

Definition

For μ be a regular cardinal smaller than κ, $\diamondsuit_\kappa^\#(cof(\mu))$ asserts the existence of a sequence $\langle N_\alpha \mid \alpha < \kappa \rangle$ such that:

1. *for every $\alpha < \kappa$, N_α is a transitive p.r.-closed set containing α, satisfying $|N_\alpha| \leqslant |\alpha| + \aleph_0$;*
2. *for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C$, $X \cap \alpha, C \cap \alpha \in N_\alpha$;*
3. *for every Π_2^1-sentence ϕ valid in a structure $\langle \kappa, \in, (A_n)_{n<\omega} \rangle$, there exists $\alpha \in cof(\mu)$, such that*

$$N_\alpha \models \text{“}\phi \text{ is valid in } \langle \alpha, \in, (A_n \restriction \alpha)_{n<\omega} \rangle.\text{”}$$

$\diamondsuit_\kappa^\#(cof(\mu))$ in L

Lemma

($V = L$) If $\kappa = \lambda^+$ is a successor cardinal and μ is a regular cardinal smaller than κ, then $\diamondsuit_\kappa^\#(cof(\mu))$ holds.

A Diamond Sequence

Proposition

Suppose $\langle N_\alpha \mid \alpha < \kappa\rangle$ *is a* $\diamondsuit_\kappa^\#(cof(\mu))$*-sequence, for some regular* $\mu < \kappa$*. Suppose that, for each infinite* $\alpha < \kappa$, $f_\alpha : \alpha \to N_\alpha$ *is a surjection. Let* $c : \kappa \times \kappa \leftrightarrow \kappa$ *be Gödel pairing function.*
For every Π^1_2*-sentence* ϕ *valid in a structure* $\langle \kappa, \in, (A_n)_{n<\omega}\rangle$*, there exists* $i < \kappa$ *such that, for every* $X \subseteq \kappa$*, for stationarily many* $\alpha < \kappa$*, the two holds:*

- $N_\alpha \models$ "ϕ *is valid in* $\langle \alpha, \in, (A_n \restriction \alpha)_{n<\omega}\rangle$";
- $X \cap \alpha = \{\beta < \alpha \mid c(i, \beta) \in f_\alpha(i)\}$.

The sets $Z^i_\alpha = \{\beta < \alpha \mid c(i, \beta) \in f_\alpha(i)\}$ witnesses $\diamondsuit_\kappa(cof(\mu))$.

Σ^1_1-completeness

Theorem

If $\diamondsuit^{\#}_{\kappa}(cof(\mu))$ holds for $\mu < \kappa$ regular, then $E^2_{\mu\text{-}club}$ is a Σ^1_1-complete equivalence relation.

Proof Suppose E is a Σ^1_1 equivalence relation. Let $i < \kappa$ be as in the previous proposition, $\mathcal{X}_\alpha$ the characteristic function of Z^i_α. For every $\eta \in 2^\kappa$ and $\alpha \in cof(\mu)$ denote by $T_{\eta\alpha}$ the set

$$\{p \in 2^\alpha \mid p \in N_\alpha \text{ and } N_\alpha \models \text{“}E \text{ is an equivalence relation and}$$

$$(p, \eta \restriction \alpha) \in E \text{ is valid in } \langle \alpha, \in, (A_n \restriction \alpha)_{n<\omega} \rangle\text{''}\}$$

$$\mathcal{F}(\eta)(\alpha) = \begin{cases} 1 & \text{if } \mathcal{X}_\alpha \in T_{\eta\alpha} \text{ and } \alpha \in cof(\mu) \\ 0 & \text{otherwise} \end{cases}$$

The Dichotomy

Theorem

Suppose $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$, $\lambda^{<\lambda} = \lambda$. If T is a theory in a countable vocabulary, and $\diamondsuit_\kappa^\#(cof(\omega))$ and $\diamondsuit_\kappa^\#(cof(\lambda))$ hold. Then one of the following holds.

- *$\cong_T$ is Δ_1^1 (all the complete extensions of T are classifiable).*
- *$\cong_T$ is Σ_1^1-complete (T has at least one non-classifiable extension).*

Questions

Question

Is there an uncountable cardinal κ, such that $H(\kappa)$ is a theorem of ZFC?

Question

Have all the non-classifiable theories the same Borel-reducibility complexity (excluding stable unsuperstable theories)?

Thank you