κ-colorable linear orders and unsuperstable theories

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Minisymposium University of Helsinki

23 February, 2022

The topology

κ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set 2^κ with the bounded topology. For every $\zeta \in 2^{<\kappa}$, the set

$$[\zeta] = \{\eta \in 2^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

Coding structures

Fix a language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in 2^\kappa$ define the structure $\mathcal{A}_f$ with domain κ and for every tuple $(a_1, a_2, \dots, a_n)$ in κ^n

$$(a_1, a_2, \dots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \dots, a_n)) > 0$$

Definition (The isomorphism relation)

Given T a first-order complete countable theory in a countable vocabulary, we say that $f, g \in 2^\kappa$ are $\cong^\kappa_T$ equivalent if
$\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
or $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

Reductions

Let E_1 and E_2 be equivalence relations on 2^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f: 2^\kappa \to 2^\kappa$ that satisfies $(x,y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.
We write $E_1 \hookrightarrow^\kappa_b E_2$.

If the function is continuous, then we say that E_1 is *continuous reducible* to E_2 and we denote it by $E_1 \hookrightarrow^\kappa_c E_2$.

Question. For any classifiable theory T and nonclassifiable theory T', is the isomorphism relation of T Borel reducible to the isomorphism relation of T'?

Equivalence modulo ω cofinality

Definition

We define the equivalence relation $=^2_\omega \subseteq 2^\kappa \times 2^\kappa$, as follows: let $S = \{\alpha < \kappa \mid cf(\alpha) = \omega\}$, $\eta =^2_\omega \xi$ if and only if $\{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\} \cap S$ contains a set that is unbounded and closed under ω-limits.

Theorem (Hyttinen - Kulikov - Moreno)

Assume T is a countable complete classifiable theory over a countable vocabulary. Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, and $\lambda^{<\lambda} = \lambda$. Then $\cong^\kappa_T \hookrightarrow^\kappa_c =^2_\omega$.

Main result

Theorem

Suppose $\kappa = \lambda^+ = 2^\lambda$ *and* $\lambda^\omega = \lambda$. *If* T *is a countable complete unsuperstable theory, then* $=^2_\omega \hookrightarrow^\kappa_c \cong_T$.

Ordered trees

Definition

Let K^{ω}_{tr} be the class of models $(A, \prec, (P_n)_{n\leq\omega}, <, h)$, where:

- there is a linear order $(I, <_I)$ such that $A \subseteq I^{\leq\omega}$;
- A is closed under initial segment;
- $\prec$ is the initial segment relation;
- $h(\eta, \xi)$ is the maximal common initial segment of η and ξ;
- let $lg(\eta)$ be the length of η (i.e. the domain of η) and $P_n = \{\eta \in A \mid lg(\eta) = n\}$ for $n \leq \omega$;

Ordered trees

Definition (continuation)

Let K^ω_{tr} be the class of models $(A, \prec, (P_n)_{n\leq\omega}, <, h)$, where:

- for every $\eta \in A$ with $lg(\eta) < \omega$, define $Suc_A(\eta)$ as $\{\xi \in A \mid \eta \prec \xi \wedge lg(\xi) = lg(\eta) + 1\}$. If $\xi < \zeta$, then there is $\eta \in A$ such that $\xi, \zeta \in Suc_A(\eta)$;
- for every $\eta \in A \backslash P_\omega$, $<\restriction Suc_A(\eta)$ is the induced linear order from I, i.e.
$$\eta^\frown\langle x\rangle < \eta^\frown\langle y\rangle \Leftrightarrow x <_I y;$$
- If η and ξ have no immediate predecessor and $\{\zeta \in A \mid \zeta \prec \eta\} = \{\zeta \in A \mid \zeta \prec \xi\}$, then $\eta = \xi$.

Coloring orders

Definition

Let I be a linear order of size κ. We say that I is κ-colorable if there is a function $F : I \to \kappa$ such that for all $B \subseteq I$, $|B| < \kappa$, $b \in I \backslash B$, and $p = tp_{bs}(b, B, I)$ such that the following hold: For all $\alpha \in \kappa$, $|\{a \in I \mid a \models p \ \& \ F(a) = \alpha\}| = \kappa$.

Theorem

Suppose I is a κ-colorable linear order. Then for any $f \in 2^\kappa$, there is an ordered coloured tree $A_f(I)$ that satisfies:
For all $f, g \in 2^\kappa$,

$$f \ =^2_\omega \ g \ \Leftrightarrow A_f(I) \cong A_g(I),$$

Initial order

Definition

Let $\mathbb{Q}$ be the linear order of the rational numbers. Let $\kappa \times \mathbb{Q}$ be ordered by the lexicographic order, I^0 be the set of functions $f : \omega \to \kappa \times \mathbb{Q}$ such that $f(n) = (f_1(n), f_2(n))$, for which $\{n \in \omega \mid f_1(n) \neq 0\}$ is finite. If $f, g \in I^0$, then $f < g$ if and only if $f(n) < g(n)$, where n is the least number such that $f(n) \neq g(n)$.

Initial order

Lemma

There is a continuous increasing sequence $\langle I^0_\alpha \mid \alpha < \kappa\rangle$ of sets of size smaller than κ, such that for all limit $\delta < \kappa$ and $\nu \in I^0$ there is $\beta < \delta$ which satisfies the following:

$$\forall \sigma \in I^0_\delta[\sigma > \nu \Rightarrow \exists \sigma' \in I^0_\beta \ (\sigma \geq \sigma' \geq \nu)]$$

In particular

There is a κ-representation $\langle I^0_\alpha \mid \alpha < \kappa\rangle$ such that for all limit $\delta < \kappa$ and $\nu \in I^0$, if $\nu \notin I^0_\delta$ there is $\beta < \delta$ which satisfies the following:

$$\forall \sigma \in I^0_\delta[\sigma > \nu \Rightarrow \exists \sigma' \in I^0_\beta \ (\sigma > \sigma' > \nu)]$$

Proof

For all $\gamma < \kappa$, let us define $\langle I^0_\alpha \mid \alpha < \kappa \rangle$ by

$$I^0_\gamma = \{\nu \in I^0 \mid \nu_1(n) < \gamma \textit{ for all } n < \omega\}$$

it is clear that $\langle I^0_\alpha \mid \alpha < \kappa \rangle$ is a κ-representation. Suppose $\delta < \kappa$ is a limit and $\nu \in I^0$. If $\nu \in I^0_\delta$, then there is $\beta < \delta$ such that $\nu \in I^0_\beta$ and the result follows. Let us take care of the case $\nu \notin I^0_\delta$. Let $\beta < \delta$ be the least ordinal such that for all $n < \omega$, $\nu_1(n) < \delta$ implies $\nu_1(n) < \beta$.

Proof

Claim: For all $\sigma \in I^0_\delta$. If $\sigma > \nu$, then there is $\sigma' \in I^0_\beta$ such that $\sigma \neq \sigma'$ and $\sigma > \sigma' > \delta$.

Proof of the claim: Let us suppose $\sigma \in I^0_\delta$ is such that $\sigma \geq \nu$. By the definition of I^0, there is $n < \omega$ such that $\sigma(n) > \nu(n)$ and n is the minimum number such that $\sigma(n) \neq \nu(n)$. Since $\sigma \in I^0_\delta$, for all $m \leq n$, $\nu_1(m) \leq \sigma_1(m) < \delta$. Thus for all $m \leq n$, $\nu_1(m) < \beta$. Let us divide the proof in two cases, $\sigma_1(n) = \nu_1(n)$ and $\sigma_1(n) > \nu_1(n)$.

Proof

Case 1. $\sigma_1(n) = \nu_1(n)$.
By the density of $\mathbb{Q}$ there is r such that $\sigma_2(n) > r > \nu_2(n)$. Let us define σ' by:

$$\sigma'(m) = \begin{cases} \nu(m) & \text{if } m < n \\ (\nu_1(n), r) & \text{if } m = n \\ 0 & \text{in other case.} \end{cases}$$

Case 2. $\sigma_1(n) > \nu_1(n)$.
Let us define σ' by:

$$\sigma'(m) = \begin{cases} \nu(m) & \text{if } m < n \\ (\nu_1(n), \nu_2(n) + 1) & \text{if } m = n \\ 0 & \text{in other case.} \end{cases}$$

Clearly $\sigma > \sigma' > \nu$. Since $\nu_1(m) < \beta$ for all $m \leq n$, $\sigma' \in I^0_\beta$.

The orders

Suppose $i < \kappa$ is such that I^i has been defined. For all $\nu \in I^i$ let ν^{i+1} be such that

$$\nu^{i+1} \models tp_{bs}(\nu, I^i \backslash \{\nu\}, I^i) \cup \{\nu > x\}.$$

Notice that ν^{i+1} is a copy of ν that is smaller than ν. Let $I^{i+1} = I^i \cup \{\nu^{i+1} \mid \nu \in I^i\}$. Suppose $i < \kappa$ is a limit ordinal such that for all $j < i$, I^j has been defined, we define I^i by $I^i = \bigcup_{j<i} I^j$.

The representations

Suppose $i < \kappa$ is such that $\langle I^i_\alpha \mid \alpha < \kappa\rangle$ has been defined. For all $\alpha < \kappa$,

$$I^{i+1}_\alpha = I^i_\alpha \cup \{\nu^{i+1} \mid \nu \in I^i_\alpha\}.$$

Suppose $i < \kappa$ is a limit ordinal such that for all $j < i$, $\langle I^j_\alpha \mid \alpha < \kappa\rangle$ has been defined, we define $\langle I^i_\alpha \mid \alpha < \kappa\rangle$ by

$$I^i_\alpha = \bigcup_{j<i} I^j_\alpha.$$

The order

Let us define I as

$$I = \bigcup_{j<\kappa} I^j$$

and the κ-representation $\langle I_\alpha \mid \alpha < \kappa \rangle$ as

$$I_\alpha = \bigcup_{\alpha<\kappa} I^\alpha_\alpha.$$

Nice property I^i

Lemma

For all $i < \kappa$, $\delta < \kappa$ a limit ordinal, and $\nu \in I^i$, there is $\beta < \delta$ that satisfies the following:

$$\forall \sigma \in I^i_\delta \; [\sigma > \nu \Rightarrow \exists \sigma' \in I^i_\beta \; (\sigma \geq \sigma' \geq \nu)]$$

In particular. If $\nu \notin I^i_\delta$ there is $\beta < \delta$ which satisfies the following:

$$\forall \sigma \in I^i_\delta [\sigma > \nu \Rightarrow \exists \sigma' \in I^0_\beta \; (\sigma > \sigma' > \nu)]$$

Nice property I

Lemma

For all $\delta < \kappa$ a limit ordinal, and $\nu \in I$, there is $\beta < \delta$ that satisfies the following:

$$\forall \sigma \in I_\delta \; [\sigma > \nu \Rightarrow \exists \sigma' \in I_\beta \; (\sigma \geq \sigma' \geq \nu)]$$

A different perspective

Definition (Generator)

For all $\nu \in I$ let us denote by $o(\nu)$ the least ordinal $\alpha < \kappa$ such that $\nu \in I^\alpha$. Let us denote the generator of ν by $Gen(\nu)$ and define it by induction as follows:

- $Gen^i(\nu) = \emptyset$, for all $i < o(\nu)$;
- $Gen^i(\nu) = \{\nu\}$, for $i = o(\nu)$;
- for all $i \geq o(\nu)$,

$$Gen^{i+1}(\nu) = Gen^i(\nu) \cup \{\sigma \in I^{i+1} \mid \exists \tau \in Gen^i(\nu)\ [\tau^{i+1} = \sigma]\};$$

- for all $i < \kappa$ limit,

$$Gen^i(\nu) = \bigcup_{j<i} Gen^j(\nu).$$

A different perspective

Finally, let

$$Gen(\nu) = \bigcup_{i<\kappa} Gen^i(\nu).$$

Suppose $\nu \in I$. For all $\sigma \in Gen(\nu)$, $\sigma \neq \nu$, there is $n < \omega$ and a sequence $\{\sigma_i\}_{i \leq n}$ such that the following holds:

- $\sigma_0 = \nu$;
- for all $j < n$,
$$\sigma_{j+1} = (\sigma_j)^{o(\sigma_{j+1})};$$
- $\sigma = \sigma_n = (\sigma_{n-1})^{o(\sigma)}$

$(<\kappa, bs)$-stable I^0

Definition

$A \in K^{\omega}_{tr}$ is $(<\kappa, bs)$-stable if for every $B \subseteq A$ of size smaller than κ,

$$\kappa > |\{tp_{bs}(a, B, A) \mid a \in A\}|.$$

Theorem (Hyttinen - Tuuri)

Let $\mathcal{R}$ be the set of functions $f : \omega \to \kappa$ for which $\{n \in \omega \mid f(n) \neq 0\}$ is finite. If $f, g \in \mathcal{R}$, then $f < g$ if and only if $f(n) < g(n)$, where n is the least number such that $f(n) \neq g(n)$. If $\lambda^{\omega} = \lambda$, then the linear order $\mathcal{R}$ is $(<\kappa, bs)$-stable.

Lemma

Suppose $\kappa = \lambda^+$ *and* $\lambda^\omega = \lambda$. I^0 *is* $(<\kappa, bs)$*-stable.*

Proof For all $A \subseteq I^0$ define $Pr(A)$ as the set $\{f_1 \mid f \in A\}$. Let $A \subseteq I^0$ be such that $|A| < \kappa$. Since $|\mathbb{Q}| = \omega$,
$|\{tp_{bs}(a, A, I^0) \mid a \in I^0\}| \leq |\{tp_{bs}(a, Pr(A), \mathcal{R}) \mid a \in \mathcal{R}\} \times 2^\omega|$.
Since $\lambda^\omega = \lambda$, $|\{tp_{bs}(a, A, I^0) \mid a \in I\}| < \kappa$.

$(<\kappa, bs)$-stable I

Lemma

Suppose $\kappa=\lambda^+$ and $\lambda^\omega=\lambda$. I is $(<\kappa, bs)$-stable.

Theorem

I is a $(<\kappa, bs)$-stable (κ, bs, bs)-nice κ-colorable linear order.

The isomorphism

Theorem (Shelah)

Suppose T is a countable complete unsuperstable theory in a countable vocabulary.
If κ is a regular uncountable cardinal, $A_1, A_2 \in K^{\omega}_{tr}$ have size κ, A_1, A_2 are locally (κ, bs, bs)-nice and $(<\kappa, bs)$-stable, $EM(A_1, \Phi)$ is isomorphic to $EM(A_2, \Phi)$, then $S(A_1) =^2_\omega S(A_2)$.

In our construction, $S(A_f(I)) =^2_\omega S(A_g(I))$ is equivalent $f \; =^2_\omega \; g$.

Thank you