On the Borel reducibility Main Gap

Miguel Moreno
University of Helsinki

Helsinki Logic Seminar
Helsinki

17 and 24 January, and 20 March, 2024

European Research Council
Established by the European Commission

Geometry

- Independence of Euclid's fifth postulate, the parallel postulate.

Geometry

- Independence of Euclid's fifth postulate, the parallel postulate.
- Khayyám (1077) and Saccheri (1733) considered the three different cases of the Khayyám-Saccheri quadrilateral (right, obtuse, and acute).

Geometry

- Independence of Euclid's fifth postulate, the parallel postulate.
- Khayyám (1077) and Saccheri (1733) considered the three different cases of the Khayyám-Saccheri quadrilateral (right, obtuse, and acute).
- Euclidean geometry, Elliptic geometry, Hyperbolic geometry.

The spectrum fuction

Let T be a countable theory over a countable language.

The spectrum fuction

Let T be a countable theory over a countable language. Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α.

The spectrum fuction

Let T be a countable theory over a countable language. Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T,\alpha)$?

Categoricity

- **1904:** Veble introduced categorical theories.

Categoricity

- **1904:** Veble introduced categorical theories.
- **1915 - 1920:** Löwenheim-Skolem Theorem.

Categoricity

- **1904:** Veble introduced categorical theories.
- **1915 - 1920:** Löwenheim-Skolem Theorem.
- **1929:** Gödel's completeness theorem.

Categoricity

- **1904:** Veble introduced categorical theories.
- **1915 - 1920:** Löwenheim-Skolem Theorem.
- **1929:** Gödel's completeness theorem.
- **1954:** Łoś and Vaught introduced κ-categorical theories.

Categoricity

- **1904:** Veble introduced categorical theories.
- **1915 - 1920:** Löwenheim-Skolem Theorem.
- **1929:** Gödel's completeness theorem.
- **1954:** Łoś and Vaught introduced κ-categorical theories.
- **1965:** Morley's categoricity theorem.

Morley's conjecture

1960's: Let T be a first-order countable theory over a countable language.

Morley's conjecture

1960's: Let T be a first-order countable theory over a countable language. For all $\aleph_0 < \lambda < \kappa$,

$$I(T,\lambda) \leq I(T,\kappa).$$

Morley's conjecture

1960's: Let T be a first-order countable theory over a countable language. For all $\aleph_0 < \lambda < \kappa$,

$$I(T,\lambda) \leq I(T,\kappa).$$

1990: Shelah proved Morley's conjecture.

Shelah's Main Gap Theorem

Theorem (Shelah 1990)

Either, for every uncountable cardinal α, $I(T,\alpha) = 2^\alpha$*; or* $\forall\alpha > 0$, $I(T,\aleph_\alpha) < \beth_{\omega_1}(\mid \alpha \mid)$.

Shelah's Main Gap Theorem

Theorem (Shelah 1990)

Either, for every uncountable cardinal α, $I(T,\alpha)=2^{\alpha}$; or $\forall\alpha>0$, $I(T,\aleph_\alpha)<\beth_{\omega_1}(\mid\alpha\mid)$.

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Descriptive Set Theory

- **1989:** Friedman and Stanley introduced the Borel reducibility between classes of countable structures.

Descriptive Set Theory

- **1989:** Friedman and Stanley introduced the Borel reducibility between classes of countable structures.

- **1993:** Mekler-Väänänen κ-separation theorem.

Descriptive Set Theory

- **1989:** Friedman and Stanley introduced the Borel reducibility between classes of countable structures.
- **1993:** Mekler-Väänänen κ-separation theorem.
- **2014:** Friedman-Hyttinen-Kulikov developed GDST and a systematic comparison between the Main Gap dividing lines and the complexity given by Borel reducibility.

The bounded topology

Let κ be an uncountable cardinal that satisfies $\kappa^{<\kappa}=\kappa$.

The bounded topology

Let κ be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

The Generalised Baire spaces

The generalised Baire space is the space κ^κ endowed with the bounded topology.

The Generalised Baire spaces

The generalised Baire space is the space κ^κ endowed with the bounded topology.

The generalised Cantor space is the subspace 2^κ.

Coding structures

Let $\omega \leq \mu \leq \kappa$ be a cardinal. Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$ and a bijection π_μ between $\mu^{<\omega}$ and μ.

Coding structures

Let $\omega \leq \mu \leq \kappa$ be a cardinal. Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$ and a bijection π_μ between $\mu^{<\omega}$ and μ.

Definition

For every $\eta \in \kappa^\kappa$ define the structure $\mathcal{A}_{\eta\restriction\mu}$ with domain μ as follows: For every tuple $(a_1, a_2, \ldots, a_n)$ in μ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_{\eta\restriction\mu}} \Leftrightarrow \eta(\pi_\mu(m, a_1, a_2, \ldots, a_n)) > 0.$$

The isomorphism relation

Definition

Let $\omega \leq \mu \leq \kappa$ be a cardinal and T a first-order theory in a relational countable language, we say that $f, g \in \kappa^\kappa$ are $\cong^\mu_T$ equivalent if one of the following holds:

The isomorphism relation

Definition

Let $\omega \leq \mu \leq \kappa$ be a cardinal and T a first-order theory in a relational countable language, we say that $f, g \in \kappa^\kappa$ are $\cong^\mu_T$ equivalent if one of the following holds:

- $\mathcal{A}_{\eta\restriction\mu} \models T, \mathcal{A}_{\xi\restriction\mu} \models T, \mathcal{A}_{\eta\restriction\mu} \cong \mathcal{A}_{\xi\restriction\mu}$

The isomorphism relation

Definition

Let $\omega \leq \mu \leq \kappa$ be a cardinal and T a first-order theory in a relational countable language, we say that $f, g \in \kappa^\kappa$ are $\cong^\mu_T$ equivalent if one of the following holds:

- $\mathcal{A}_{\eta\restriction\mu} \models T, \mathcal{A}_{\xi\restriction\mu} \models T, \mathcal{A}_{\eta\restriction\mu} \cong \mathcal{A}_{\xi\restriction\mu}$
- $\mathcal{A}_{\eta\restriction\mu} \not\models T, \mathcal{A}_{\xi\restriction\mu} \not\models T$

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ.

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *reducible* to E_2, if there is a function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *reducible* to E_2, if there is a function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$. We write $E_1 \hookrightarrow_r E_2$.

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *reducible* to E_2, if there is a function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x,y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$. We write $E_1 \hookrightarrow_r E_2$.

We can define a partial order on the set of all first-order complete countable theories

$$T \leq^\kappa T' \ \textit{iff} \ \cong_T \hookrightarrow_C \cong_{T'}$$

Non-classifiable theories

A theory T is non-classifiable if it is a countable complete theory that satisfies one of the following:

- T is unstable;

Non-classifiable theories

A theory T is non-classifiable if it is a countable complete theory that satisfies one of the following:

- T is unstable;
- T is stable unsuperstable;

Non-classifiable theories

A theory T is non-classifiable if it is a countable complete theory that satisfies one of the following:

- T is unstable;
- T is stable unsuperstable;
- T is superstable with DOP;

Non-classifiable theories

A theory T is non-classifiable if it is a countable complete theory that satisfies one of the following:

- T is unstable;
- T is stable unsuperstable;
- T is superstable with DOP;
- T is superstable with OTOP.

Classifiable theories

Classifiable are divided into:

- shallow,

$$I(T, \aleph_\alpha) < \beth_{\omega_1}(|\alpha|);$$

Classifiable theories

Classifiable are divided into:

- shallow,

$$I(T,\aleph_\alpha) < \beth_{\omega_1}(|\alpha|);$$

- non-shallow,

$$I(T,\alpha) = 2^\alpha.$$

First dividing lines

Fact (Friedman-Hyttinen-Kulikov 2014)

1. *Let $\kappa^{<\kappa} = \kappa > 2^\omega$. If T is classifiable and shallow, then $\cong_T$ is κ-Borel.*

First dividing lines

Fact (Friedman-Hyttinen-Kulikov 2014)

1. *Let $\kappa^{<\kappa} = \kappa > 2^\omega$. If T is classifiable and shallow, then $\cong_T$ is κ-Borel.*
2. *If T is classifiable non-shallow, then $\cong_T$ is $\Delta_1^1(\kappa)$ not κ-Borel.*

First dividing lines

Fact (Friedman-Hyttinen-Kulikov 2014)

1. *Let* $\kappa^{<\kappa} = \kappa > 2^\omega$. *If* T *is classifiable and shallow, then* $\cong_T$ *is* κ*-Borel.*
2. *If* T *is classifiable non-shallow, then* $\cong_T$ *is* $\Delta_1^1(\kappa)$ *not* κ*-Borel.*
3. *If* T *is unstable or stable with the OTOP or superstable with the DOP and* $\kappa > \omega_1$, *then* $\cong_T$ *is not* $\Delta_1^1(\kappa)$.

First dividing lines

Fact (Friedman-Hyttinen-Kulikov 2014)

1. *Let $\kappa^{<\kappa} = \kappa > 2^\omega$. If T is classifiable and shallow, then $\cong_T$ is κ-Borel.*
2. *If T is classifiable non-shallow, then $\cong_T$ is $\Delta^1_1(\kappa)$ not κ-Borel.*
3. *If T is unstable or stable with the OTOP or superstable with the DOP and $\kappa > \omega_1$, then $\cong_T$ is not $\Delta^1_1(\kappa)$.*
4. *If T is stable unsuperstable, then $\cong_T$ is not κ-Borel.*

Question

Question: What can we say about the Borel-reducibility between different dividing lines?

Classifiable and shallow

Theorem (Mangraviti - Motto Ros 2020)

Let κ be such that $\kappa > 2^\omega$. If T is classifiable and shallow with depth α, then $rk_B(\cong_T) \leq 4\alpha$.

Classifiable and shallow

Theorem (Mangraviti - Motto Ros 2020)

Let κ be such that $\kappa > 2^\omega$. If T is classifiable and shallow with depth α, then $rk_B(\cong_T) \leq 4\alpha$.

Theorem (Mangraviti - Motto Ros 2020)

Let $\kappa = \aleph_\gamma$ be such that $\kappa^{<\kappa} = \kappa$ and $\beth_{\omega_1}(|\gamma|) \leq \kappa$. Let T, T' be countable complete first-order theories, and suppose T is classifiable and shallow, while T' is not. Then

$$\cong_T \hookrightarrow_B \cong_{T'}$$

General reduction

Fact (Mangraviti-Motto Ros)

Let E_1 be a Borel equivalence relation with $\gamma \leq \kappa$ equivalence classes and E_2 be an equivalence relation with θ equivalence classes. If $\gamma \leq \theta$, then $E_1 \hookrightarrow_B E_2$.

Counting α-classes relation

Let $\alpha < \kappa$ be an ordinal and $0 < \varrho \leq \kappa$. $\eta\ \alpha_\varrho\ \xi$ if and only if one of the following holds:

Counting α-classes relation

Let $\alpha < \kappa$ be an ordinal and $0 < \varrho \leq \kappa$. $\eta\ \alpha_\varrho\ \xi$ if and only if one of the following holds:

- ϱ is finite:
  - $\eta(\alpha) = \xi(\alpha) < \varrho - 1$;
  - $\eta(\alpha), \xi(\alpha) \geq \varrho - 1$.

Counting α-classes relation

Let $\alpha < \kappa$ be an ordinal and $0 < \varrho \leq \kappa$. $\eta \; \alpha_\varrho \; \xi$ if and only if one of the following holds:

- ϱ is finite:
  - $\eta(\alpha) = \xi(\alpha) < \varrho - 1$;
  - $\eta(\alpha), \xi(\alpha) \geq \varrho - 1$.
- ϱ is infinite:
  - $\eta(\alpha) = \xi(\alpha) < \varrho$;
  - $\eta(\alpha), \xi(\alpha) \geq \varrho$.

Few equivalence classes

Lemma (M. 2023)

Suppose $\kappa > 2^\omega$ and T is a countable first-order theory in a countable vocabulary (not necessarily complete) such that $\cong_T$ has $\varrho \leq \kappa$ equivalence classes. Then for all $\alpha < \kappa$

$$\cong_T \hookrightarrow_B \alpha_\varrho \text{ and } \alpha_\varrho \hookrightarrow_L \cong_T .$$

Few equivalence classes

Lemma (M. 2023)

Suppose $\kappa > 2^\omega$ and T is a countable first-order theory in a countable vocabulary (not necessarily complete) such that $\cong_T$ has $\varrho \leq \kappa$ equivalence classes. Then for all $\alpha < \kappa$

$$\cong_T \hookrightarrow_B \ \alpha_\varrho \text{ and } \ \alpha_\varrho \ \hookrightarrow_L \cong_T .$$

Even more, if T is not categorical then $\cong_T \not\hookrightarrow_C \ \alpha_\varrho$.

Proof

- $\cong_T \hookrightarrow_B \alpha_\varrho$ follows from Mangraviti-Motto Ros.

Proof

- $\cong_T \hookrightarrow_B \alpha_\varrho$ follows from Mangraviti-Motto Ros.
- $\eta \restriction \alpha + 1$ determines the equivalence class of η. So $\alpha_\varrho \hookrightarrow_L \cong_T$.

Proof

- $\cong_T \hookrightarrow_B \alpha_\varrho$ follows from Mangraviti-Motto Ros.
- $\eta \restriction \alpha+1$ determines the equivalence class of η. So $\alpha_\varrho \hookrightarrow_L \cong_T$.
- α_ϱ is open, so $\cong_T \hookrightarrow_C \alpha_\varrho$ implies $\cong_T$ is open.

Proof

- $\cong_T \hookrightarrow_B \ \alpha_\varrho$ follows from Mangraviti-Motto Ros.
- $\eta \restriction \alpha + 1$ determines the equivalence class of η. So $\alpha_\varrho \ \hookrightarrow_L \cong_T$.
- α_ϱ is open, so $\cong_T \hookrightarrow_C \ \alpha_\varrho$ implies $\cong_T$ is open.
- $\cong_T$ is open iff T is categorical (Mangraviti-Motto Ros), so if T is not categorical then $\cong_T \not\hookrightarrow_C \ \alpha_\varrho$.

Gap: Shallow and Non-shallow

Theorem (M. 2023)

Suppose $\aleph_\mu = \kappa = \lambda^+ = 2^\lambda$ *is such that* $\beth_{\omega_1}(|\mu|) \leq \kappa$.

Gap: Shallow and Non-shallow

Theorem (M. 2023)

Suppose $\aleph_\mu = \kappa = \lambda^+ = 2^\lambda$ *is such that* $\beth_{\omega_1}(|\mu|) \leq \kappa$. *Let* T_1 *be a countable complete classifiable shallow theory with* $\varrho = I(\kappa, T_1)$, T_2 *be a countable complete theory not classifiable shallow.*

Gap: Shallow and Non-shallow

Theorem (M. 2023)

Suppose $\aleph_\mu = \kappa = \lambda^+ = 2^\lambda$ *is such that* $\beth_{\omega_1}(|\mu|) \leq \kappa$. *Let* T_1 *be a countable complete classifiable shallow theory with* $\varrho = I(\kappa, T_1)$, T_2 *be a countable complete theory not classifiable shallow. If* T *is classifiable shallow such that* $1 < I(\kappa, T) < I(\kappa, T_1)$, *then*

$$\cong_T \hookrightarrow_B 0_\varrho \hookrightarrow_L \cong_{T_1} \hookrightarrow_B 0_\kappa \hookrightarrow_L \cong_{T_2}.$$

Gap: Shallow and Non-shallow

Theorem (M. 2023)

Suppose $\aleph_\mu = \kappa = \lambda^+ = 2^\lambda$ *is such that* $\beth_{\omega_1}(|\mu|) \leq \kappa$. *Let* T_1 *be a countable complete classifiable shallow theory with* $\varrho = I(\kappa, T_1)$, T_2 *be a countable complete theory not classifiable shallow. If* T *is classifiable shallow such that* $1 < I(\kappa, T) < I(\kappa, T_1)$, *then*

$$\cong_T \hookrightarrow_B \ 0_\varrho \ \hookrightarrow_L \cong_{T_1} \hookrightarrow_B \ 0_\kappa \ \hookrightarrow_L \cong_{T_2} .$$

In particular

$$\cong_{T_2} \not\hookrightarrow_r \ 0_\kappa \ \not\hookrightarrow_r \cong_{T_1} \not\hookrightarrow_C \ 0_\varrho \ \not\hookrightarrow_r \cong_T .$$

Consistency

Theorem (Hyttinen - Kulikov - M. 2017)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, and $\lambda^{<\lambda} = \lambda$. There is a κ-closed κ^+-cc forcing which forces:

Consistency

Theorem (Hyttinen - Kulikov - M. 2017)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, and $\lambda^{<\lambda} = \lambda$. There is a κ-closed κ^+-cc forcing which forces: If T is classifiable and T' is non-classifiable, then $T \leq^\kappa T'$ and $T' \not\leq^\kappa T$.

Unsuperstable theories

Theorem (Hyttinen - Kulikov - M. 2017)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, and $\lambda^\omega = \lambda$. If T is classifiable and T' is stable unsuperstable, then $T \leq^\kappa T'$ and $T' \not\leq^\kappa T$.

Unsuperstable theories

Theorem (Hyttinen - Kulikov - M. 2017)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, *and* $\lambda^\omega = \lambda$. *If* T *is classifiable and* T' *is stable unsuperstable, then* $T \leq^\kappa T'$ *and* $T' \not\leq^\kappa T$.

Theorem (M. 2023)

Suppose $\kappa = \lambda^+ = 2^\lambda$ *and* $\lambda^\omega = \lambda$. *If* T *is a classifiable theory, and* T' *is an unsuperstable theory, then* $T \leq^\kappa T'$ *and* $T' \not\leq^\kappa T$.

Equivalence modulo γ cofinality

Definition

We define the equivalence relation $=^2_\gamma \subseteq 2^\kappa \times 2^\kappa$, as follows: let $S = \{\alpha < \kappa \mid cf(\alpha) = \gamma\}$,

$$\eta =^2_\gamma \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

Borel-reducibility Main Gap

Theorem (M. 2023)

Let $\mathfrak{c} = 2^{\omega}$. *Suppose* $\kappa = \lambda^{+} = 2^{\lambda}$ *and* $2^{\mathfrak{c}} \leq \lambda = \lambda^{\omega_1}$.

Borel-reducibility Main Gap

Theorem (M. 2023)

Let $\mathfrak{c} = 2^\omega$. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $2^\mathfrak{c} \leq \lambda = \lambda^{\omega_1}$. If T is a classifiable theory, and T' is a non-classifiable theory, then there is $\gamma < \kappa$ such that

$$\cong_T \hookrightarrow_C =^2_\gamma \hookrightarrow_C \cong_{T'} \text{ and } =^2_\gamma \not\hookrightarrow_B \cong_T .$$

Borel-reducibility Main Gap

Theorem (M. 2023)

Let $\mathfrak{c} = 2^\omega$. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{\omega_1}$. If T is a classifiable theory, and T' is a non-classifiable theory, then there is $\gamma < \kappa$ such that

$$\cong_T \hookrightarrow_C =^2_\gamma \hookrightarrow_C \cong_{T'} \text{ and } =^2_\gamma \not\hookrightarrow_B \cong_T .$$

In particular

$$T \leq^\kappa T' \text{ and } T' \not\leq^\kappa T.$$

Classifiable theories

Theorem (Hyttinen - Kulikov - M. 2017)

Assume T is a classifiable theory and let $S = \{\alpha < \kappa \mid cf(\alpha) = \gamma\}$. If $\diamondsuit_S$ holds, then $\cong_T \hookrightarrow_C =^2_\gamma$.

The reductions

Theorem (M. 2023)

Let κ be inaccessible or $\kappa = \lambda^+ = 2^\lambda$. Suppose T is a non-classifiable theory.

1. *If T is stable unsuperstable, then let $\theta = \gamma = \omega$.*

The reductions

Theorem (M. 2023)

Let κ be inaccessible or $\kappa = \lambda^+ = 2^\lambda$. Suppose T is a non-classifiable theory.

1. *If T is stable unsuperstable, then let $\theta = \gamma = \omega$.*
2. *If T is unstable, or superstable with OTOP, then let $\theta = \omega$ and $\omega \leq \gamma < \kappa$.*

The reductions

Theorem (M. 2023)

Let κ be inaccessible or $\kappa = \lambda^+ = 2^\lambda$. Suppose T is a non-classifiable theory.

1. *If T is stable unsuperstable, then let $\theta = \gamma = \omega$.*
2. *If T is unstable, or superstable with OTOP, then let $\theta = \omega$ and $\omega \leq \gamma < \kappa$.*
3. *If T is superstable with DOP, then let $\theta = 2^\omega = \mathfrak{c}$ and $\omega_1 \leq \gamma < \kappa$.*

The reductions

Theorem (M. 2023)

Let κ be inaccessible or $\kappa = \lambda^+ = 2^\lambda$. Suppose T is a non-classifiable theory.

1. *If T is stable unsuperstable, then let $\theta = \gamma = \omega$.*
2. *If T is unstable, or superstable with OTOP, then let $\theta = \omega$ and $\omega \leq \gamma < \kappa$.*
3. *If T is superstable with DOP, then let $\theta = 2^\omega = \mathfrak{c}$ and $\omega_1 \leq \gamma < \kappa$.*

If θ, γ, and κ satisfy that $\forall \alpha < \kappa$, $\alpha^\gamma < \kappa$, and $(2^\theta)^+ \leq \kappa$, then

$$=^2_\gamma \hookrightarrow_C \cong_T .$$

Blue print of the proof

- Construct an ε-dense, (κ, ε)-nice, $(< \kappa)$-stable, and κ-colorable linear order.

Blue print of the proof

- Construct an ε-dense, (κ, ε)-nice, $(< \kappa)$-stable, and κ-colorable linear order.
- Construct ordered trees from the linear order.

Blue print of the proof

- Construct an ε-dense, (κ, ε)-nice, $(< \kappa)$-stable, and κ-colorable linear order.
- Construct ordered trees from the linear order.
- Construct skeletons from ordered trees, to construct Ehrenfeucht-Mostowski models.

Blue print of the proof

- Construct an ε-dense, (κ, ε)-nice, $(< \kappa)$-stable, and κ-colorable linear order.
- Construct ordered trees from the linear order.
- Construct skeletons from ordered trees, to construct Ehrenfeucht-Mostowski models.
- Prove the isomorphism theorem.

Blue print of the proof

- Construct an ε-dense, (κ, ε)-nice, $(< \kappa)$-stable, and κ-colorable linear order.
- Construct ordered trees from the linear order.
- Construct skeletons from ordered trees, to construct Ehrenfeucht-Mostowski models.
- Prove the isomorphism theorem.
- Construct the reductions.

ε-dense

Definition

Let I be a linear order of size κ and ε a regular cardinal smaller than κ. We say that I is ε*-dense* if the following holds.

ε-dense

Definition

Let I be a linear order of size κ and ε a regular cardinal smaller than κ. We say that I is *ε-dense* if the following holds.

If $A, B \subseteq I$ are subsets of size less than ε such that for all $a \in A$ and $b \in B$, $a < b$, then there is $c \in I$, such that for all $a \in A$ and $b \in B$, $a < c < b$.

κ-representation

Definition

Let A be an arbitrary set of size κ.

κ-representation

Definition

Let A be an arbitrary set of size κ. The sequence $\mathbb{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ is a *κ-representation* of A, if $\langle A_\alpha \mid \alpha < \kappa \rangle$ is an increasing continuous sequence of subsets of A, for all $\alpha < \kappa$, $|A_\alpha| < \kappa$, and $\bigcup_{\alpha<\kappa} A_\alpha = A$.

(κ, ε)-nice

Definition

Let $\varepsilon < \kappa$ be a regular cardinal, A be a linear order of size κ and $\langle A_\alpha \mid \alpha < \kappa \rangle$ a κ-representation.

(κ, ε)-nice

Definition

Let $\varepsilon < \kappa$ be a regular cardinal, A be a linear order of size κ and $\langle A_\alpha \mid \alpha < \kappa \rangle$ a κ-representation. Then A is (κ, ε)-*nice* if there is a club $C \subseteq \kappa$, such that for all limit $\delta \in C$ with $cf(\delta) \geq \varepsilon$, for all $x \in A$ there is $\beta < \delta$ such that one of the following holds:

- $\forall \sigma \in A_\delta [\sigma \geq x \Rightarrow \exists \sigma' \in A_\beta \ (\sigma \geq \sigma' \geq x)]$
- $\forall \sigma \in A_\delta [\sigma \leq x \Rightarrow \exists \sigma' \in A_\beta \ (\sigma \leq \sigma' \leq x)]$

$(<\kappa)$-stable

Definition

A linear order I is $(<\kappa)$-*stable* if for every $B \subseteq I$ of size smaller than κ,

$$\kappa > |\{tp_{bs}(a, B, I) \mid a \in I\}|.$$

κ-colorable

Definition

Let I be a linear order of size κ. We say that I is *κ-colorable* if there is a function $F : I \to \kappa$ such that for all $B \subseteq I$, $|B| < \kappa$, $b \in I \backslash B$, and $p = tp_{bs}(b, B, I)$ such that the following hold: For all $\alpha \in \kappa$,

$$|\{a \in I \mid a \models p \ \& \ F(a) = \alpha\}| = \kappa.$$

Hyttinen - Tuuri's order

Definition (Hyttinen - Tuuri 1991)

Let $\mathcal{R}$ be the set of functions $f : \omega \to \kappa$, for which $|\{n \in \omega \mid f(\alpha) \neq 0\}|$ is finite.

Hyttinen - Tuuri's order

Definition (Hyttinen - Tuuri 1991)

Let $\mathcal{R}$ be the set of functions $f : \omega \to \kappa$, for which $|\{n \in \omega \mid f(\alpha) \neq 0\}|$ is finite.
If $f, g \in \mathcal{R}$, then $f < g$ if and only if $f(n) < g(n)$, where n is the least number such that $f(n) \neq g(n)$.

Hyttinen - Tuuri's order

Definition (Hyttinen - Tuuri 1991)

Let $\mathcal{R}$ be the set of functions $f : \omega \to \kappa$, for which $|\{n \in \omega \mid f(\alpha) \neq 0\}|$ is finite.
If $f, g \in \mathcal{R}$, then $f < g$ if and only if $f(n) < g(n)$, where n is the least number such that $f(n) \neq g(n)$.

Fact (Hyttinen-Tuuri 1991)

The linear order $\mathcal{R}$ is (κ, ω)-nice and $(< \kappa)$-stable.

The F^{φ}_{ω} isolation

Definition

Let $\varphi(x,y) :=$ "$y > x$", we define F^{φ}_{ω} in the following way. Let $|B| < \kappa$ and $p \in S_{bs}(B)$, $(p, A) \in F^{\varphi}_{\omega}$ if and only if $A \subseteq B$, A is finite, and there is $a \in A$ such that

$$\{a > x, x = a\} \cap p \neq \emptyset \;\&\; a \models p \restriction B \backslash \{a\}.$$

F^{φ}_{ω}-construction

Definition

A sequence $(A,(a_i,B_i)_{i<\alpha})$ is an F^{φ}_{ω}-construction over A if for all $i<\alpha$, $(tp_{bs}(a_i,A_i),B_i)\in F^{\varphi}_{\omega}$ where $A_i=A\cup\bigcup_{j<i}a_j$.

F^{φ}_{ω}-construction

Definition

A sequence $(A, (a_i, B_i)_{i<\alpha})$ is an F^{φ}_{ω}-construction over A if for all $i < \alpha$, $(tp_{bs}(a_i, A_i), B_i) \in F^{\varphi}_{\omega}$ where $A_i = A \cup \bigcup_{j<i} a_j$.
C is F^{φ}_{ω}-constructible over A if there is an F^{φ}_{ω}-construction over A such that $C = A \cup \bigcup_{j<\alpha} a_j$.

$(F^{\varphi}_{\omega}, \kappa)$-primary

Definition

C is $(F^{\varphi}_{\omega}, \kappa)$-saturated if for all $B \subseteq C$ of size smaller than κ, and $p \in S_{bs}(B)$, $(p, A) \in F^{\varphi}_{\omega}$ implies that p is realized in C.

$(F^{\varphi}_{\omega}, \kappa)$-primary

Definition

C is $(F^{\varphi}_{\omega}, \kappa)$-saturated if for all $B \subseteq C$ of size smaller than κ, and $p \in S_{bs}(B)$, $(p, A) \in F^{\varphi}_{\omega}$ implies that p is realized in C.

Definition

C is $(F^{\varphi}_{\omega}, \kappa)$-primary over A if it is F^{φ}_{ω}-constructible over A and $(F^{\varphi}_{\omega}, \kappa)$-saturated.

$(F^{\varphi}_{\omega}, \kappa)$-primary

Lemma (M. 2023)

There is an $(F^{\varphi}_{\omega}, \kappa)$-primary over $\mathcal{R}$

$(F^{\varphi}_{\omega}, \kappa)$-primary

Lemma (M. 2023)

There is an $(F^{\varphi}_{\omega}, \kappa)$-primary over $\mathcal{R}$ and it is an ω-dense, (κ, ω)-nice, $(< \kappa)$-stable, and κ-colorable linear order.

Existence

Let $\theta < \kappa$ be the smallest cardinal such that there is a ε-dense model of DLO of size θ.

Existence

Let $\theta < \kappa$ be the smallest cardinal such that there is a ε-dense model of *DLO* of size θ.

Theorem (M. 2023)

Suppose κ is inaccessible, or $\kappa = \lambda^+$, $2^\theta \leq \lambda = \lambda^{<\varepsilon}$. There is a ε-dense, (κ, ε)-nice, $(< \kappa)$-stable, and κ-colorable linear order.

Construction

Let $\mathcal{Q}$ be a model of DLO of size $\theta < \kappa$, that is ε-dense.

Definition

Let $\kappa \times \mathcal{Q}$ be ordered by the lexicographic order,

Construction

Let $\mathcal{Q}$ be a model of DLO of size $\theta < \kappa$, that is ε-dense.

Definition

Let $\kappa \times \mathcal{Q}$ be ordered by the lexicographic order, $\mathcal{I}^0$ be the set of functions $f : \varepsilon \to \kappa \times \mathcal{Q}$ such that $f(\alpha) = (f_1(\alpha), f_2(\alpha))$, for which $|\{\alpha \in \varepsilon \mid f_1(\alpha) \neq 0\}|$ is smaller than ε.

Construction

Let $\mathcal{Q}$ be a model of *DLO* of size $\theta < \kappa$, that is ε-dense.

Definition

Let $\kappa \times \mathcal{Q}$ be ordered by the lexicographic order, $\mathcal{I}^0$ be the set of functions $f : \varepsilon \to \kappa \times \mathcal{Q}$ such that $f(\alpha) = (f_1(\alpha), f_2(\alpha))$, for which $|\{\alpha \in \varepsilon \mid f_1(\alpha) \neq 0\}|$ is smaller than ε.
If $f, g \in \mathcal{I}^0$, then $f < g$ if and only if $f(\alpha) < g(\alpha)$, where α is the least number such that $f(\alpha) \neq g(\alpha)$.

Construction

Let us fix $\tau \in \mathcal{Q}$. Let I be the set of functions $f : \varepsilon \to (\{0\} \times \mathcal{I}^0) \cup (\kappa \times \mathcal{Q})$ such that the following hold:

Construction

Let us fix $\tau \in \mathcal{Q}$. Let I be the set of functions $f : \varepsilon \to (\{0\} \times \mathcal{I}^0) \cup (\kappa \times \mathcal{Q})$ such that the following hold:

- $f \restriction \{0\} : \{0\} \to \{0\} \times \mathcal{I}^0$.

Construction

Let us fix $\tau \in \mathcal{Q}$. Let I be the set of functions $f : \varepsilon \to (\{0\} \times \mathcal{I}^0) \cup (\kappa \times \mathcal{Q})$ such that the following hold:

- $f \restriction \{0\} : \{0\} \to \{0\} \times \mathcal{I}^0$.
- $f \restriction \varepsilon \backslash \{0\} : \varepsilon \backslash \{0\} \to \kappa \times \mathcal{Q}$.

Construction

Let us fix $\tau \in \mathcal{Q}$. Let I be the set of functions $f : \varepsilon \to (\{0\} \times \mathcal{I}^0) \cup (\kappa \times \mathcal{Q})$ such that the following hold:

- $f \restriction \{0\} : \{0\} \to \{0\} \times \mathcal{I}^0$.
- $f \restriction \varepsilon \backslash \{0\} : \varepsilon \backslash \{0\} \to \kappa \times \mathcal{Q}$.
- There is $\alpha < \varepsilon$ ordinal such that $\forall \beta > \alpha$, $f(\beta) = (0, \tau)$. We say that the least α with such property is the *depth* of f and we denote it by $dp(f)$;

Construction

Let us fix $\tau \in \mathcal{Q}$. Let I be the set of functions $f : \varepsilon \to (\{0\} \times \mathcal{I}^0) \cup (\kappa \times \mathcal{Q})$ such that the following hold:

- $f \restriction \{0\} : \{0\} \to \{0\} \times \mathcal{I}^0$.
- $f \restriction \varepsilon \backslash \{0\} : \varepsilon \backslash \{0\} \to \kappa \times \mathcal{Q}$.
- There is $\alpha < \varepsilon$ ordinal such that $\forall \beta > \alpha$, $f(\beta) = (0, \tau)$. We say that the least α with such property is the *depth* of f and we denote it by $dp(f)$;
- There are functions $f_1 : \varepsilon \to \kappa$ and $f_2 : \varepsilon \to \mathcal{I}^0 \cup \mathcal{Q}$ such that $f(\beta) = (f_1(\beta), f_2(\beta))$ and $f_1 \restriction dp(f) + 1$ is strictly increasing.

Construction

We say that $f < g$ if and only if one of the following holds:

Construction

We say that $f < g$ if and only if one of the following holds:

- $f(0) \neq g(0)$ and $f_2(0) < g_2(0)$;

Construction

We say that $f < g$ if and only if one of the following holds:

- $f(0) \neq g(0)$ and $f_2(0) < g_2(0)$;
- let $\alpha = dp(g)$, $\forall \beta \leq \alpha$, $f(\beta) = g(\beta)$ and $f_1(\alpha + 1) \neq 0$;

Construction

We say that $f < g$ if and only if one of the following holds:

- $f(0) \neq g(0)$ and $f_2(0) < g_2(0)$;
- let $\alpha = dp(g)$, $\forall \beta \leq \alpha$, $f(\beta) = g(\beta)$ and $f_1(\alpha + 1) \neq 0$;
- exists $\alpha > 0$ such that $\forall \beta < \alpha$, $f(\beta) = g(\beta)$, and $f_1(\alpha), g_1(\alpha) \neq 0$ and $g(\alpha) > f(\alpha)$.

Generators

Definition

For all $f \in I$ with depth α, define the *generator* of f, $Gen(f)$, by

$$Gen(f) = \{g \in I \mid f \restriction \alpha + 1 = g \restriction \alpha + 1\}.$$

Generators

- If $f \neq g$ and $g \in Gen(f)$, then $f > g$.

Generators

- If $f \neq g$ and $g \in Gen(f)$, then $f > g$.
- Let $f \in Gen(\nu)$. If $g \notin Gen(\nu)$, then $g < \nu$ if and only if $g < f$.

Generators

- If $f \neq g$ and $g \in Gen(f)$, then $f > g$.
- Let $f \in Gen(\nu)$. If $g \notin Gen(\nu)$, then $g < \nu$ if and only if $g < f$.
- If $f \in Gen(\nu)$

$$f \models tp_{bs}(\nu, I \setminus Gen(\nu), I) \cup \{\nu > x\}.$$

Generators

- If $f \neq g$ and $g \in Gen(f)$, then $f > g$.
- Let $f \in Gen(\nu)$. If $g \notin Gen(\nu)$, then $g < \nu$ if and only if $g < f$.
- If $f \in Gen(\nu)$

$$f \models tp_{bs}(\nu, I \backslash Gen(\nu), I) \cup \{\nu > x\}.$$

- Let $f \in Gen(\nu)$. If $\sigma \in I$ is such that $\nu \geq \sigma \geq f$, then $\sigma \in Gen(\nu)$.

Iterations

For all $f \in I$ with depth α, define $o(f) = f_1(\alpha)$ the *complexity* of f.

Iterations

For all $f \in I$ with depth α, define $o(f) = f_1(\alpha)$ the *complexity* of f.

Let

$$I^0 = \{f \in I \mid f : \varepsilon \to (\{0\} \times \mathcal{I}^0) \cup (\{0\} \times \{\tau\})\}.$$

Iterations

For all $f \in I$ with depth α, define $o(f) = f_1(\alpha)$ the *complexity* of f.

Let

$$I^0 = \{f \in I \mid f : \varepsilon \to (\{0\} \times \mathcal{I}^0) \cup (\{0\} \times \{\tau\})\}.$$

Suppose i is such that I^i is defined. Let

$$I^{i+1} = \{f \in I \mid o(f) \leq i+1\}.$$

Iterations

For all $f \in I$ with depth α, define $o(f) = f_1(\alpha)$ the *complexity* of f.

Let

$$I^0 = \{f \in I \mid f : \varepsilon \to (\{0\} \times \mathcal{I}^0) \cup (\{0\} \times \{\tau\})\}.$$

Suppose i is such that I^i is defined. Let

$$I^{i+1} = \{f \in I \mid o(f) \leq i+1\}.$$

Suppose i is a limit ordinal such that for all $j < i$, I^j is defined, let

$$I^i = \bigcup_{j<i} I^j.$$

κ-representation

Define $\langle \mathcal{I}^0_\alpha \mid \alpha < \kappa \rangle$ by

$$\mathcal{I}^0_\alpha = \{\nu \in \mathcal{I}^0 \mid \nu_1(n) < \alpha \textit{ for all } n < \varepsilon\},$$

and $\langle I^0_\alpha \mid \alpha < \kappa \rangle$ in the canonical way.

κ-representation

Define $\langle \mathcal{I}^0_\alpha \mid \alpha < \kappa \rangle$ by

$$\mathcal{I}^0_\alpha = \{\nu \in \mathcal{I}^0 \mid \nu_1(n) < \alpha \textit{ for all } n < \varepsilon\},$$

and $\langle I^0_\alpha \mid \alpha < \kappa \rangle$ in the canonical way.

Suppose $i < \kappa$ is such that $\langle I^i_\alpha \mid \alpha < \kappa \rangle$ has been defined. For all $\alpha < \kappa$ let

$$I^{i+1}_\alpha = \{f \in I \mid o(f) \leq i+1 \ \& \ f_2(0) \in I^0_\alpha\},$$

κ-representation

Define $\langle \mathcal{I}^0_\alpha \mid \alpha < \kappa\rangle$ by

$$\mathcal{I}^0_\alpha = \{\nu \in \mathcal{I}^0 \mid \nu_1(n) < \alpha \textit{ for all } n < \varepsilon\},$$

and $\langle I^0_\alpha \mid \alpha < \kappa\rangle$ in the canonical way.

Suppose $i < \kappa$ is such that $\langle I^i_\alpha \mid \alpha < \kappa\rangle$ has been defined. For all $\alpha < \kappa$ let

$$I^{i+1}_\alpha = \{f \in I \mid o(f) \leq i + 1 \ \&\ f_2(0) \in I^0_\alpha\},$$

for $i < \kappa$ is a limit ordinal so

$$I^i_\alpha = \bigcup_{j<i} I^j_\alpha.$$

κ-representation

Let $\nu \in I^i_\delta$. For all $f \in Gen(\nu)$, $f \in I^{o(f)}_\delta$.

κ-representation

Let $\nu \in I^i_\delta$. For all $f \in Gen(\nu)$, $f \in I^{o(f)}_\delta$.

Let us define the κ-representation $\langle I_\alpha \mid \alpha < \kappa \rangle$ by

$$I_\alpha = I^\alpha_\alpha.$$

Roads

Definition

For all $\nu \in I$ with $dp(\nu) = \alpha$, there is a maximal sequence $\langle \nu_i \mid i \leq \alpha \rangle$ such that $\nu_0 \in I^0$, $\nu_\alpha = \nu$, and for all $i < j$, $\nu_i \in Gen(\nu_i)$.

We call this sequence *the road from I^0 to ν*.

Roads

Definition

For all $\nu \in I$ with $dp(\nu) = \alpha$, there is a maximal sequence $\langle \nu_i \mid i \leq \alpha \rangle$ such that $\nu_0 \in I^0$, $\nu_\alpha = \nu$, and for all $i < j$, $\nu_i \in Gen(\nu_i)$.

We call this sequence *the road from I^0 to ν*.

Fact

Let $\langle \nu_j \mid j \leq \alpha \rangle$ be the road from I^0 to ν_α. For all $i < \alpha$

$$\nu_\alpha \models tp_{bs}(\nu_i, I^{o(\nu_{i+1})} \backslash (Gen(\nu_{i+1}) \cup \{\nu_i\}), I) \cup \{\nu_i > x\}$$

The properties

Theorem (M. 2023)

Suppose κ *is inaccessible, or* $\kappa = \lambda^+$, $2^\theta \leq \lambda = \lambda^{<\varepsilon}$. *Then I is* ε*-dense,* $(<\kappa)$*-stable,* (κ, ε)*-nice, and* κ*-colorable.*

Proof

- I is ε-dense.

Proof

- I is ε-dense.
- For all $\delta < \kappa$ limit with $cf(\delta) \geq \varepsilon$, and $\nu \in I$, there is $\beta < \delta$ that satisfies the following:

$$\forall \sigma \in I_\delta \ [\sigma > \nu \Rightarrow \exists \sigma' \in I_\beta \ (\sigma \geq \sigma' \geq \nu)]$$

Proof

- I is ε-dense.
- For all $\delta < \kappa$ limit with $cf(\delta) \geq \varepsilon$, and $\nu \in I$, there is $\beta < \delta$ that satisfies the following:

$$\forall \sigma \in I_\delta \; [\sigma > \nu \Rightarrow \exists \sigma' \in I_\beta \; (\sigma \geq \sigma' \geq \nu)]$$

- Suppose $\kappa = \lambda^+$ and $2^\theta \leq \lambda = \lambda^{<\varepsilon}$. I is $(< \kappa)$-stable.

Proof

- I is ε-dense.

- For all $\delta < \kappa$ limit with $cf(\delta) \geq \varepsilon$, and $\nu \in I$, there is $\beta < \delta$ that satisfies the following:

$$\forall \sigma \in I_\delta \; [\sigma > \nu \Rightarrow \exists \sigma' \in I_\beta \; (\sigma \geq \sigma' \geq \nu)]$$

- Suppose $\kappa = \lambda^+$ and $2^\theta \leq \lambda = \lambda^{<\varepsilon}$. I is $(< \kappa)$-stable.

- I is a κ-colorable linear order.

κ^+, $(\gamma+2)$-tree*

Let $\gamma < \kappa$ be a regular cardinal. A κ^+, $(\gamma+2)$-tree* t is a tree with the following properties:

- t has a unique root.

κ^+, $(\gamma + 2)$-tree*

Let $\gamma < \kappa$ be a regular cardinal. A κ^+, $(\gamma + 2)$-tree* t is a tree with the following properties:

- t has a unique root.
- Every element of t has less than κ^+ immediate successors.

κ^+, $(\gamma + 2)$-tree*

Let $\gamma < \kappa$ be a regular cardinal. A κ^+, $(\gamma + 2)$-tree* t is a tree with the following properties:

- t has a unique root.
- Every element of t has less than κ^+ immediate successors.
- All the branches of t have order type γ or $\gamma + 1$.

κ^+, $(\gamma + 2)$-tree*

Let $\gamma < \kappa$ be a regular cardinal. A κ^+, $(\gamma + 2)$-tree* t is a tree with the following properties:

- t has a unique root.
- Every element of t has less than κ^+ immediate successors.
- All the branches of t have order type γ or $\gamma + 1$.
- Every chain of length less than γ has a unique limit.

Isomorphism of κ^+, $(\gamma+2)$-tree*

Lemma (Hyttinen - Kulikov - M.)

Suppose $\gamma<\kappa$ is such that for all $\epsilon<\kappa$, $\epsilon^\gamma<\kappa$. For every $f,g\in 2^\kappa$ there are κ^+, $(\gamma+2)$-trees J_f and J_g such that*

$$f \ =^2_\gamma \ g \Leftrightarrow J_f \cong_{ct} J_g$$

where $\cong_{ct}$ is the isomorphism of κ^+, $(\gamma+2)$-tree.*

Ordered trees

Definition

Let $\gamma < \kappa$ be a regular cardinal and I a linear order. $(A, \prec, <)$ is an ordered tree if the following holds:

- $(A, \prec)$ is a κ^+, $(\gamma + 2)$-tree*.

Ordered trees

Definition

Let $\gamma < \kappa$ be a regular cardinal and I a linear order. $(A, \prec, <)$ is an ordered tree if the following holds:

- $(A, \prec)$ is a κ^+, $(\gamma + 2)$-tree*.
- for all $x \in A$, $(succ(x), <)$ is isomorphic to I.

Isomorphism of ordered trees

Theorem (M. 2023)

Suppose $\gamma < \kappa$ is such that for all $\epsilon < \kappa$, $\epsilon^\gamma < \kappa$, and there is a κ-colorable linear order I.

Isomorphism of ordered trees

Theorem (M. 2023)

Suppose $\gamma < \kappa$ is such that for all $\epsilon < \kappa$, $\epsilon^\gamma < \kappa$, and there is a κ-colorable linear order I. For all $f \in 2^\kappa$ there is an ordered tree A_f such that for all $f, g \in 2^\kappa$,

$$f \ =^2_\gamma \ g \ \Leftrightarrow A_f \cong A_g.$$

κ-colorable

Definition

Let I be a linear order of size κ. We say that I is *κ-colorable* if there is a function $F : I \to \kappa$ such that for all $B \subseteq I$, $|B| < \kappa$, $b \in I \setminus B$, and $p = tp_{bs}(b, B, I)$ such that the following hold: For all $\alpha \in \kappa$,

$$|\{a \in I \mid a \models p \ \& \ F(a) = \alpha\}| = \kappa.$$

The models

Suppose κ is inaccessible, or $\kappa = \lambda^+$, $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. Let $\gamma < \kappa$ be such that for all $\epsilon < \kappa$, $\epsilon^\gamma < \kappa$.

The models

Suppose κ is inaccessible, or $\kappa = \lambda^+$, $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. Let $\gamma < \kappa$ be such that for all $\epsilon < \kappa$, $\epsilon^\gamma < \kappa$.

Lemma

Suppose T is superstable with DOP in a countable relational vocabulary τ. Let τ^1 be a Skolemization of τ, and T^1 be a complete theory in τ^1 extending T and with Skolem-functions in τ. Then for every $f \in 2^\kappa$ there is $\mathcal{M}_1^f \models T^1$ with the following properties.

The models

Lemma

1. *There is a map* $\mathcal{H} : A_f \to (\text{dom } \mathcal{M}_1^f)^n$ *for some* $n < \omega$, $\eta \mapsto a_\eta$, *such that* $\mathcal{M}_1^f$ *is the Skolem hull of* $\{a_\eta \mid \eta \in A_f\}$. *Let us denote* $\{a_\eta \mid \eta \in A_f\}$ *by* $Sk(\mathcal{M}_1^f)$.

The models

Lemma

1. *There is a map* $\mathcal{H} : A_f \to (\text{dom } \mathcal{M}_1^f)^n$ *for some* $n < \omega$, $\eta \mapsto a_\eta$, *such that* $\mathcal{M}_1^f$ *is the Skolem hull of* $\{a_\eta \mid \eta \in A_f\}$. *Let us denote* $\{a_\eta \mid \eta \in A_f\}$ *by* $Sk(\mathcal{M}_1^f)$.
2. $\mathcal{M}^f = \mathcal{M}_1^f \restriction \tau$ *is a model of* T.

The models

Lemma

1. *There is a map* $\mathcal{H} : A_f \to (\text{dom } \mathcal{M}_1^f)^n$ *for some* $n < \omega$, $\eta \mapsto a_\eta$, *such that* $\mathcal{M}_1^f$ *is the Skolem hull of* $\{a_\eta \mid \eta \in A_f\}$. *Let us denote* $\{a_\eta \mid \eta \in A_f\}$ *by* $Sk(\mathcal{M}_1^f)$.
2. $\mathcal{M}^f = \mathcal{M}_1^f \restriction \tau$ *is a model of* T.
3. $Sk(\mathcal{M}_1^f)$ *is indiscernible in* $\mathcal{M}_1^f$ *relative to* $L_{\omega_1\omega_1}$, *i.e. if* $tp_{at}(\bar{s}, \emptyset, A_f) = tp_{at}(\bar{s'}, \emptyset, A_f)$, *then* $tp_\Delta(\bar{a}_{\bar{s}}, \emptyset, \mathcal{M}_1^f) = tp_\Delta(\bar{a}_{\bar{s'}}, \emptyset, \mathcal{M}_1^f)$, *where* $\Delta = L_{\omega_1\omega_1}$.

The models

Lemma

1. *There is a map* $\mathcal{H} : A_f \to (\text{dom } \mathcal{M}_1^f)^n$ *for some* $n < \omega$, $\eta \mapsto a_\eta$, *such that* $\mathcal{M}_1^f$ *is the Skolem hull of* $\{a_\eta \mid \eta \in A_f\}$. *Let us denote* $\{a_\eta \mid \eta \in A_f\}$ *by* $Sk(\mathcal{M}_1^f)$.
2. $\mathcal{M}^f = \mathcal{M}_1^f \restriction \tau$ *is a model of* T.
3. $Sk(\mathcal{M}_1^f)$ *is indiscernible in* $\mathcal{M}_1^f$ *relative to* $L_{\omega_1\omega_1}$, *i.e. if* $tp_{at}(\bar{s}, \emptyset, A_f) = tp_{at}(\bar{s'}, \emptyset, A_f)$, *then* $tp_\Delta(\bar{a}_{\bar{s}}, \emptyset, \mathcal{M}_1^f) = tp_\Delta(\bar{a}_{\bar{s'}}, \emptyset, \mathcal{M}_1^f)$, *where* $\Delta = L_{\omega_1\omega_1}$.
4. *There is a formula* $\varphi \in L_{\omega_1\omega_1}(\tau)$ *such that for all* $\eta, \nu \in A_f$ *and* $m < \gamma$, *if* $A_f \models P_m(\eta) \wedge P_\gamma(\nu)$, *then* $\mathcal{M}^f \models \varphi(a_\nu, a_\eta)$ *if and only if* $A_f \models \eta \prec \nu$.

Coding trees

For every $f \in 2^\kappa$ let us define the order $K^D(f)$ by:

I. $dom\ K^D(f) = (dom\ A_f \times \{0\}) \cup (dom\ A_f \times \{1\})$.

Coding trees

For every $f \in 2^\kappa$ let us define the order $K^D(f)$ by:

I. $dom\ K^D(f) = (dom\ A_f \times \{0\}) \cup (dom\ A_f \times \{1\})$.

II. For all $\eta \in A_f$, $(\eta, 0) <_{K^D(f)} (\eta, 1)$.

Coding trees

For every $f \in 2^\kappa$ let us define the order $K^D(f)$ by:

I. $dom\ K^D(f) = (dom\ A_f \times \{0\}) \cup (dom\ A_f \times \{1\})$.

II. For all $\eta \in A_f$, $(\eta, 0) <_{K^D(f)} (\eta, 1)$.

III. If $\eta, \xi \in A_f$, then $\eta \prec \xi$ if and only if

$$(\eta, 0) <_{K^D(f)} (\xi, 0) <_{K^D(f)} (\xi, 1) <_{K^D(f)} (\eta, 1).$$

Coding trees

For every $f \in 2^\kappa$ let us define the order $K^D(f)$ by:

I. $dom\ K^D(f) = (dom\ A_f \times \{0\}) \cup (dom\ A_f \times \{1\})$.

II. For all $\eta \in A_f$, $(\eta, 0) <_{K^D(f)} (\eta, 1)$.

III. If $\eta, \xi \in A_f$, then $\eta \prec \xi$ if and only if

$$(\eta, 0) <_{K^D(f)} (\xi, 0) <_{K^D(f)} (\xi, 1) <_{K^D(f)} (\eta, 1).$$

IV. If $\eta, \xi \in A_f$, then $\eta < \xi$ if and only if $(\eta, 1) <_{K^D(f)} (\xi, 0)$.

EM-models

Lemma (Shelah, Hyttinen-Tuuri)

Suppose T is a countable superstable theory with the DOP in a countable vocabulary τ. Then there exists a vocabulary $\tau^1 \supseteq \tau$, $|\tau^1| = \omega_1$, such that for every linear order I we can find a τ^1-model $\mathcal{N}$ which is an Ehrenfeucht-Mostowski model of T for I, where the order is definable by an $L_{\omega_1\omega_1}$-formula.

Homogenicity

Let $\upsilon \leq \kappa$ be a regular cardinal, a tree *A* is υ*-homogeneous with respect to quantifier free formulas* if the following holds:

Homogenicity

Let $\upsilon \leq \kappa$ be a regular cardinal, a tree A is *υ-homogeneous with respect to quantifier free formulas* if the following holds: *For every partial isomorphisms $F : X \to A$, where $|X| < \varepsilon$ is a subset of A, and a in A; there is a partial isomorphisms $g : X \cup a \to A$ that extends F.*

Homogenicity

Let $\upsilon \leq \kappa$ be a regular cardinal, a tree A is *υ-homogeneous with respect to quantifier free formulas* if the following holds: *For every partial isomorphisms $F : X \to A$, where $|X| < \varepsilon$ is a subset of A, and a in A; there is a partial isomorphisms $g : X \cup a \to A$ that extends F.*

Fact

For all $f \in 2^{\kappa}$, A_f is ε-homogeneous with respect to quantifier free formulas.

Homogenicity

Let $\upsilon \leq \kappa$ be a regular cardinal, a tree A is *υ-homogeneous with respect to quantifier free formulas* if the following holds: *For every partial isomorphisms $F : X \to A$, where $|X| < \varepsilon$ is a subset of A, and a in A; there is a partial isomorphisms $g : X \cup a \to A$ that extends F.*

Fact

For all $f \in 2^\kappa$, A_f is ε-homogeneous with respect to quantifier free formulas.

Fact

If $tp_{at}(\bar{s}, \emptyset, A_f) = tp_{at}(\bar{s'}, \emptyset, A_f)$, then $tp_\Delta(\bar{a}_{\bar{s}}, \emptyset, \mathcal{M}_1^f) = tp_\Delta(\bar{a}_{\bar{s}'}, \emptyset, \mathcal{M}_1^f)$, where $\Delta = L_{\omega_1\omega_1}$.

The isomorphism theorem

Theorem (M. 2023)

Suppose T is a non-classifiable first order theory in a countable relational vocabulary τ.

1. *If T is unstable or superstable with OTOP, $\omega \leq \gamma < \kappa$ is such that for all $\alpha < \kappa$, $\alpha^\gamma < \kappa$, then for all $f, g \in 2^\kappa$*

$$f \ =^2_\gamma \ g \text{ iff } \mathcal{M}^f \cong \mathcal{M}^g.$$

The isomorphism theorem

Theorem (M. 2023)

Suppose T is a non-classifiable first order theory in a countable relational vocabulary τ.

1. *If T is unstable or superstable with OTOP, $\omega \leq \gamma < \kappa$ is such that for all $\alpha < \kappa$, $\alpha^{\gamma} < \kappa$, then for all $f, g \in 2^{\kappa}$*
$$f =^2_{\gamma} g \text{ iff } \mathcal{M}^f \cong \mathcal{M}^g.$$
2. *If T is superstable with DOP, κ is inaccessible or $\kappa = \lambda^+$ and $2^{\mathfrak{c}} \leq \lambda$, and $\omega_1 \leq \gamma < \kappa$ is such that for all $\alpha < \kappa$, $\alpha^{\gamma} < \kappa$, then for all $f, g \in 2^{\kappa}$,*
$$f =^2_{\gamma} g \text{ iff } \mathcal{M}^f \cong \mathcal{M}^g.$$

Proof

There is a κ-representation $\mathbb{A} = \{(A_f)_\alpha\}_{\alpha<\kappa}$ and $B^f(\eta, \alpha) = Succ_{a_F}(\eta) \cap (A_f)_\alpha$.

Proof

There is a κ-representation $\mathbb{A} = \{(A_f)_\alpha\}_{\alpha<\kappa}$ and $B^f(\eta,\alpha) = Succ_{a_F}(\eta) \cap (A_f)_\alpha$. Such that there is a club C^f for all $\delta \in C^f$ with $cf(\delta) \geq \varepsilon$, $\eta \in A_f$, $A_f \not\models P_\gamma(\eta)$, and $\nu \in Suc_{A_f}(\eta)$,

$$\forall \sigma \in B^f(\eta,\delta)\ [\sigma > \nu \Rightarrow \exists \sigma' \in B^f(\eta,\beta)\ (\sigma \geq \sigma' \geq \nu)].$$

Proof

Suppose $f, g \in 2^\kappa$ are such that $f \neq^2_\gamma g$, and $\mathcal{M}^f$ are $\mathcal{M}^g$ isomorphic. Let $F : \mathcal{M}^f \to \mathcal{M}^g$ be an isomorphism between $\mathcal{M}^f$ and $\mathcal{M}^g$.

Proof

Suppose $f, g \in 2^\kappa$ are such that $f \neq^2_\gamma g$, and $\mathcal{M}^f$ are $\mathcal{M}^g$ isomorphic. Let $F : \mathcal{M}^f \to \mathcal{M}^g$ be an isomorphism between $\mathcal{M}^f$ and $\mathcal{M}^g$. Let us denote by $\bar{a}_\eta$ and $\bar{b}_\xi$ the elements of $Sk(\mathcal{M}^f_1)$ and $Sk(\mathcal{M}^g_1)$.

Proof

Suppose $f, g \in 2^\kappa$ are such that $f \neq^2_\gamma g$, and $\mathcal{M}^f$ are $\mathcal{M}^g$ isomorphic. Let $F : \mathcal{M}^f \to \mathcal{M}^g$ be an isomorphism between $\mathcal{M}^f$ and $\mathcal{M}^g$. Let us denote by $\bar{a}_\eta$ and $\bar{b}_\xi$ the elements of $Sk(\mathcal{M}^f_1)$ and $Sk(\mathcal{M}^g_1)$. For each $\eta \in A_f$ let

$$F(a_\eta) = (\mu^0_\eta(\bar{b}_{\bar{v}_\eta}), \dots, \mu^m_\eta(\bar{b}_{\bar{v}_\eta})) = \bar{\mu}_\eta(\bar{b}_{\bar{v}_\eta}),$$

Proof

Let $\bar{v}_\eta = (v^i_\eta)_{i<lg(\bar{v}_\eta)}$. Let

- $C_1 = \{\delta \in C_0 \mid \forall \eta \in A_f \ (\eta \in (A_f)_\delta$ implies $\bar{v}_\eta \subseteq (A_g)_\delta)\}$;

Proof

Let $\bar{v}_\eta = (v^i_\eta)_{i<lg(\bar{v}_\eta)}$. Let

- $C_1 = \{\delta \in C_0 \mid \forall \eta \in A_f \ (\eta \in (A_f)_\delta \text{ implies } \bar{v}_\eta \subseteq (A_g)_\delta)\}$;
- $C'_2 = \{\delta \in C_1 \mid \forall \alpha < \delta \ \forall \eta \in (A_f)_\delta \ \forall \sigma_1 \in B^f(\eta,\kappa) \ \exists \sigma_2 \in B^f(\eta,\delta)$

  $[\bar{v}_{\sigma_1}, \bar{v}_{\sigma_2}$ realizes the same atomic type over $(A_g)_\alpha$ and $\bar{\mu}_{\sigma_1} = \bar{\mu}_{\sigma_2}]\}$

Proof

Let $\bar{v}_\eta = (v^i_\eta)_{i<lg(\bar{v}_\eta)}$. Let

- $C_1 = \{\delta \in C_0 \mid \forall \eta \in A_f \ (\eta \in (A_f)_\delta \text{ implies } \bar{v}_\eta \subseteq (A_g)_\delta)\}$;
- $C'_2 = \{\delta \in C_1 \mid \forall \alpha < \delta \ \forall \eta \in (A_f)_\delta \ \forall \sigma_1 \in B^f(\eta, \kappa) \ \exists \sigma_2 \in B^f(\eta, \delta)$

  $[\bar{v}_{\sigma_1}, \bar{v}_{\sigma_2}$ realizes the same atomic type over $(A_g)_\alpha$ and $\bar{\mu}_{\sigma_1} = \bar{\mu}_{\sigma_2}]\}$
- $C_2 = \{\delta \in C'_2 \mid cf(\delta) \geq \gamma\}$

Proof

Let $\bar{v}_\eta = (v^i_\eta)_{i<lg(\bar{v}_\eta)}$. Let

- $C_1 = \{\delta \in C_0 \mid \forall \eta \in A_f \ (\eta \in (A_f)_\delta$ implies $\bar{v}_\eta \subseteq (A_g)_\delta)\}$;
- $C'_2 = \{\delta \in C_1 \mid \forall \alpha < \delta \ \forall \eta \in (A_f)_\delta \ \forall \sigma_1 \in B^f(\eta, \kappa) \ \exists \sigma_2 \in B^f(\eta, \delta)$

  $[\bar{v}_{\sigma_1}, \bar{v}_{\sigma_2}$ realizes the same atomic type over $(A_g)_\alpha$ and $\bar{\mu}_{\sigma_1} = \bar{\mu}_{\sigma_2}]\}$

- $C_2 = \{\delta \in C'_2 \mid cf(\delta) \geq \gamma\}$
- $C = \{\delta \in C_2 \mid \delta \in C_2$ & δ is a limit point of $C_2\}$.

Proof

Let $\delta \in S \cap C$, so there is $\eta \in A_f$, such that:

1. $A_f \models P_\gamma(\eta)$.

Proof

Let $\delta \in S \cap C$, so there is $\eta \in A_f$, such that:

1. $A_f \models P_\gamma(\eta)$.
2. For all $n < \gamma$, $\eta \restriction n \in (A_f)_\delta$.

Proof

Let $\delta \in S \cap C$, so there is $\eta \in A_f$, such that:

1. $A_f \models P_\gamma(\eta)$.
2. For all $n < \gamma$, $\eta \restriction n \in (A_f)_\delta$.
3. For all $\alpha < \delta$, there is $m < \gamma$ such that $\eta \restriction m \notin (A_f)_\alpha$.

Proof

For each $n < lg(\bar{v}_\eta)$ there is $\alpha_n \in C_2 \cap \delta$ such that one of the following holds

I. $v^n_\eta \in (A_g)_{\alpha_n}$.

Proof

For each $n < lg(\bar{v}_\eta)$ there is $\alpha_n \in C_2 \cap \delta$ such that one of the following holds

I. $v_\eta^n \in (A_g)_{\alpha_n}$.

II. There is $m_n < lg(v_\eta^n)$ such that for $w^0 = v_\eta^n \restriction m_n$ and $w^1 = v_\eta^n \restriction (m_n + 1)$ the following hold
   - $w^0 \in (A_g)_{\alpha_n}$ and $w^1 \notin (A_g)_\delta$.

Proof

For each $n < lg(\bar{v}_\eta)$ there is $\alpha_n \in C_2 \cap \delta$ such that one of the following holds

I. $v^n_\eta \in (A_g)_{\alpha_n}$.

II. There is $m_n < lg(v^n_\eta)$ such that for $w^0 = v^n_\eta \restriction m_n$ and $w^1 = v^n_\eta \restriction (m_n + 1)$ the following hold
   - $w^0 \in (A_g)_{\alpha_n}$ and $w^1 \notin (A_g)_\delta$.
   - $\forall \sigma \in B^g(w^0, \delta)\ [\sigma > w^1 \Rightarrow \exists \sigma' \in B^g(w^0, \alpha_n)\ (\sigma \geq \sigma' \geq w^1)]$.

Proof

$$tp_L(\bar{b}_{\bar{v}_{\zeta_1}^\frown \bar{v}_\eta}, \emptyset, \mathcal{M}) = tp_L(\bar{b}_{\bar{v}_{\zeta_2}^\frown \bar{v}_\eta}, \emptyset, \mathcal{M})$$

Proof

$$tp_L(\bar{b}_{\bar{v}_{\zeta_1}\frown\bar{v}_\eta}, \emptyset, \mathcal{M}) = tp_L(\bar{b}_{\bar{v}_{\zeta_2}\frown\bar{v}_\eta}, \emptyset, \mathcal{M})$$

Since $\bar{\mu}_{\zeta_1} = \bar{\mu}_{\zeta_2}$,

$$\mathcal{M}_1^g \models \varphi(\bar{\mu}_\eta(\bar{b}_{\bar{v}_\eta}), \bar{\mu}_{\zeta_1}(\bar{b}_{\bar{v}_{\zeta_1}})) \Leftrightarrow \varphi(\bar{\mu}_\eta(\bar{b}_{\bar{v}_\eta}), \bar{\mu}_{\zeta_2}(\bar{b}_{\bar{v}_{\zeta_2}}))$$

Proof

$$tp_L(\bar{b}_{\bar{v}_{\zeta_1}\frown\bar{v}_\eta}, \emptyset, \mathcal{M}) = tp_L(\bar{b}_{\bar{v}_{\zeta_2}\frown\bar{v}_\eta}, \emptyset, \mathcal{M})$$

Since $\bar{\mu}_{\zeta_1} = \bar{\mu}_{\zeta_2}$,

$$\mathcal{M}_1^g \models \varphi(\bar{\mu}_\eta(\bar{b}_{\bar{v}_\eta}), \bar{\mu}_{\zeta_1}(\bar{b}_{\bar{v}_{\zeta_1}})) \Leftrightarrow \varphi(\bar{\mu}_\eta(\bar{b}_{\bar{v}_\eta}), \bar{\mu}_{\zeta_2}(\bar{b}_{\bar{v}_{\zeta_2}}))$$

so

$$\mathcal{M}_1^f \models \varphi(\bar{a}_\eta, \bar{a}_{\zeta_1}) \Leftrightarrow \varphi(\bar{a}_\eta, \bar{a}_{\zeta_2}).$$

Proof

$$tp_L(\bar{b}_{\bar{v}_{\zeta_1}\frown\bar{v}_\eta}, \emptyset, \mathcal{M}) = tp_L(\bar{b}_{\bar{v}_{\zeta_2}\frown\bar{v}_\eta}, \emptyset, \mathcal{M})$$

Since $\bar{\mu}_{\zeta_1} = \bar{\mu}_{\zeta_2}$,

$$\mathcal{M}_1^g \models \varphi(\bar{\mu}_\eta(\bar{b}_{\bar{v}_\eta}), \bar{\mu}_{\zeta_1}(\bar{b}_{\bar{v}_{\zeta_1}})) \Leftrightarrow \varphi(\bar{\mu}_\eta(\bar{b}_{\bar{v}_\eta}), \bar{\mu}_{\zeta_2}(\bar{b}_{\bar{v}_{\zeta_2}}))$$

so

$$\mathcal{M}_1^f \models \varphi(\bar{a}_\eta, \bar{a}_{\zeta_1}) \Leftrightarrow \varphi(\bar{a}_\eta, \bar{a}_{\zeta_2}).$$

On the other hand, since $\zeta_1 \prec \eta$ and $\zeta_2 \not\prec \eta$,

Proof

$$tp_L(\bar{b}_{\bar{v}_{\zeta_1}^\frown \bar{v}_\eta}, \emptyset, \mathcal{M}) = tp_L(\bar{b}_{\bar{v}_{\zeta_2}^\frown \bar{v}_\eta}, \emptyset, \mathcal{M})$$

Since $\bar{\mu}_{\zeta_1} = \bar{\mu}_{\zeta_2}$,

$$\mathcal{M}_1^g \models \varphi(\bar{\mu}_\eta(\bar{b}_{\bar{v}_\eta}), \bar{\mu}_{\zeta_1}(\bar{b}_{\bar{v}_{\zeta_1}})) \Leftrightarrow \varphi(\bar{\mu}_\eta(\bar{b}_{\bar{v}_\eta}), \bar{\mu}_{\zeta_2}(\bar{b}_{\bar{v}_{\zeta_2}}))$$

so

$$\mathcal{M}_1^f \models \varphi(\bar{a}_\eta, \bar{a}_{\zeta_1}) \Leftrightarrow \varphi(\bar{a}_\eta, \bar{a}_{\zeta_2}).$$

On the other hand, since $\zeta_1 \prec \eta$ and $\zeta_2 \not\prec \eta$,

$$\mathcal{M}^f \models \varphi(\bar{a}_\eta, \bar{a}_{\zeta_1}) \wedge \neg\varphi(\bar{a}_\eta, \bar{a}_{\zeta_2}),$$

a contradiction, since $\mathcal{M}^f = \mathcal{M}_1^f \restriction \tau$ and $\varphi \in L(\tau)$.

Stable unsuperstable theories

Fact (M. 2023)

If T is a countable complete stable unsuperstable theory over a countable vocabulary, then for all $f, g \in 2^\kappa$, $f \ =^2_\omega \ g$ if and only if $EM(A_f, \Phi)$ and $EM(A_g, \Phi)$ are isomorphic.

$\cong_T \hookrightarrow_C =^2_\mu$, $\kappa = \lambda^+$

Theory	$\lambda = \lambda^\gamma$	$\diamondsuit_\lambda$	$\mathrm{DI}^*_{S^\kappa_\gamma}(\Pi^1_1)$
Classifiable	$\omega \leq \mu \leq \gamma$	$\mu = \lambda$	$\mu = \gamma$
Non-classifiable	Indep	Indep	$\mu = \gamma$

$=^2_\mu \hookrightarrow_C \cong_T$, $\kappa = \lambda^+$

Theory	$\lambda = \lambda^\gamma$	$2^{\mathfrak{c}} \leq \lambda = \lambda^\gamma$	$2^{\mathfrak{c}} \leq \lambda = \lambda^{<\lambda}$ & $\diamondsuit_\lambda$
Stable Unsuperstable	$\mu = \omega$	$\mu = \omega$	$\mu = \omega$
Unstable	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \lambda$
Superstable with OTOP	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \lambda$
Superstable with DOP	?	$\omega_1 \leq \mu \leq \gamma$	$\omega_1 \leq \mu \leq \lambda$

A bigger Gap

Theorem (M. 2023)

Suppose κ is inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{\omega_1}$. There exists a cofinality-preserving forcing extension in which the following holds:

A bigger Gap

Theorem (M. 2023)

Suppose κ is inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{\omega_1}$. There exists a cofinality-preserving forcing extension in which the following holds:
If T_1 is classifiable and T_2 is not. Then there is a regular cardinal $\gamma < \kappa$ such that, if $X, Y \subseteq S^\kappa_\gamma$ are stationary and disjoint, then $=^2_X$ and $=^2_Y$ are strictly in between $\cong_{T_1}$ and $\cong_{T_2}$.

Main Gap Dichotomy

Theorem (M. 2023)

Let κ be inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. There exists a $<\kappa$-closed κ^+-cc forcing extension in which for any countable first-order theory in a countable vocabulary (not necessarily complete), T, one of the following holds:

Main Gap Dichotomy

Theorem (M. 2023)

Let κ be inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. There exists a $< \kappa$-closed κ^+-cc forcing extension in which for any countable first-order theory in a countable vocabulary (not necessarily complete), T, one of the following holds:

- $\cong_T$ *is* $\Delta^1_1(\kappa)$;
- $\cong_T$ *is* $\Sigma^1_1(\kappa)$*-complete.*

Non-classifiable theories

Lemma (M. 2023)

Let κ *be strongly inaccessible, or* $\kappa = \lambda^+ = 2^\lambda$ *and* $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. *For all cardinals* $\aleph_0 < \mu < \delta < \kappa$, *if* T *is a non-classifiable theory then*

$$\cong^{\mu}_{T} \hookrightarrow_C \cong^{\delta}_{T} \hookrightarrow_C \ id \ \hookrightarrow_C \cong_T .$$

Classifiable non-shallow

Lemma (M. 2023)

Suppose $\kappa = \lambda^+ = 2^\lambda$. The following reduction is strict. Let $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. If T_1 is a classifiable non-shallow theory and T_2 is a non-classifiable theory, then

$$\cong^{\lambda}_{T_2} \hookrightarrow_C \cong_{T_1} \hookrightarrow_C \cong_{T_2}.$$

Classifiable shallow

Lemma (M. 2023)

Suppose $\kappa = \lambda^+ = 2^\lambda$*. The following reductions are strict. Let* $\kappa = \aleph_\gamma$ *be such that* $\beth_{\omega_1}(|\gamma|) \leq \kappa$*. Suppose* T_1 *is a classifiable shallow theory,* T_2 *a classifiable non-shallow theory, and* T_3 *non-classifiable theory. Then*

$$\cong_{T_1} \hookrightarrow_B \cong^\lambda_{T_3} \hookrightarrow_C \cong_{T_2} .$$

Thank you

Article at: https://arxiv.org/abs/2308.07510