Filter Reflection and Generalised Descriptive Set Theory

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Helsinki University Logic Seminar

October 2020

This is a joint work with Gabriel Fernandes and Assaf Rinot at BIU.

Our paper, entitled **Fake Reflection** is available at
`https://arxiv.org/abs/2003.08340`

Outline

Outline

Stationary reflection

Let α be an ordinal of uncountable cofinality. A set $C \subseteq \alpha$ is a club if it is closed and unbounded. A set $S \subseteq \alpha$ is stationary if for all club $C \subset \alpha$, $C \cap S \neq \emptyset$.

Definition

Let κ be a regular uncountable cardinal $\alpha \in \kappa$ be an ordinal of uncountable cofinality, and a stationary $S \subseteq \kappa$, we say that S reflects at α if $S \cap \alpha$ is stationary in α

If κ is a weakly compact cardinal, every stationary subset of κ reflects at a regular cardinal $\alpha < \kappa$.

Generalised descriptive set theory

Suppose κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

The generalised Baire space is the space κ^κ endowed with the bounded topology, for every $\eta \in \kappa^{<\kappa}$ the following set

$$N_\eta = \{\xi \in \kappa^\kappa \mid \eta \subseteq \xi\}$$

is a basic open set.

Equivalence modulo nonstationary

Definition

For every stationary set $S \subseteq \kappa$ and $\theta \in [2, \kappa]$, the equivalence relation $=^\theta_S$ over the subspace θ^κ is defined via

$$\eta =^\theta_S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\} \text{ is non-stationary.}$$

Definition

The quasi-order $\leq^S$ over κ^κ is defined via

$$\eta \leq^S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \text{ is non-stationary.}$$

The quasi-order $\subseteq^S$ over 2^κ is nothing but $\leq^S \cap (2^\kappa \times 2^\kappa)$.

Model Theory and $=^\theta_S$

In model theory, Shelah's main gap theorem can be understood as: *Classifiable theories are less complex than non-classifiable theories.* In generalized descriptive set theory, the complexity of a theory can be study by studying the complexity of the isomorphism relation of the theory. Let λ be a regular cardinal and denote by S^κ_λ the set $\{\alpha<\kappa \mid cf(\alpha)=\lambda\}$. Let us denote by $=^\theta_\lambda$ the relation $=^\theta_S$ when $S=S^\kappa_\lambda$.

Fact (Hyttinen-M)

The isomorphism relation of any classifiable theory is less complex than $=^\kappa_\lambda$ for all λ.

Under some cardinal arithmetic assumptions the following can be proved:

Fact (Friedman-Hyttinen-Kulikov)

Suppose T is a non-classifiable theory. There is a regular cardinal $\lambda<\kappa$ such that $=^2_\lambda$ is as most as complex as the isomorphism relation of T.

Reductions

For $i < 2$, let X_i be some space from the collection $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$. Let R_0 and R_1 be binary relations over X_0 and X_1, respectively.

Definition

A function $f : X_0 \to X_1$ is said to be a reduction of R_0 to R_1 iff, for all $\eta, \xi \in X_0$,

$$\eta \; R_0 \; \xi \text{ iff } f(\eta) \; R_1 \; f(\xi).$$

The existence of a function f satisfying this is denoted by $R_0 \hookrightarrow R_1$.

Lipschitz reductions

For $i < 2$, let X_i be some space from the collection $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$. Let R_0 and R_1 be binary relations over X_0 and X_1, respectively.

For all $\eta, \xi \in \kappa^\kappa$, denote

$$\Delta(\eta, \xi) := \min(\{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cup \{\kappa\}).$$

A reduction f of R_0 to R_1 is said to be 1-Lipschitz iff for all $\eta, \xi \in X_0$,

$$\Delta(\eta, \xi) \leq \Delta(f(\eta), f(\xi)).$$

The existence of a 1-Lipschitz reduction f is denoted by $R_0 \hookrightarrow_1 R_1$. We likewise define $R_0 \hookrightarrow_c R_1$, $R_0 \hookrightarrow_B R_1$ and $R_0 \hookrightarrow_{BM} R_1$ once we replace 1-Lipschitz by a continuous, Borel, or Baire measurable map, respectively.

Comparing $=^\kappa_S$ and $=^2_S$

Fact (Asperó-Hyttinen-Kulikov-M)

If every stationary subset of X reflects at stationary many $\alpha \in Y$, then $=^\kappa_X \hookrightarrow_c =^\kappa_Y$.

Fact (Friedman-Hyttinen-Kulikov)

Suppose $V = L$, and $X \subseteq \kappa$ and $Y \subseteq reg(\kappa)$ are stationary. If every stationary subset of X reflects at stationary many $\alpha \in Y$, then $=^2_X \hookrightarrow_c =^2_Y$.

Limitations

Let λ be a regular cardinal and denote by S^κ_λ the set $\{\alpha < \kappa \mid cf(\alpha) = \lambda\}$.

- For all regular cardinals $\gamma \leqslant \lambda < \kappa$, any $X \subseteq S^\kappa_\lambda$, X does not reflect at any $\alpha \in S^\kappa_\gamma$.
- If $\kappa = \lambda^+$ and $\square_\lambda$ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that Y does not reflect at any $\alpha < \kappa$. This happens in L.
- Usual stationary reflection requires large cardinals.

Outline

The case of L

Recall: $=^\theta_\lambda$ is the relation $=^\theta_S$ when $S = S^\kappa_\lambda$.

Fact (Hyttinen-Kulikov-M)

Suppose $V = L$. Let λ be a regular cardinal below κ. Then for all stationary $X \subseteq \kappa$, $=^\kappa_X \hookrightarrow_c =^2_\lambda$.

Question

How is this possible if there are sets in L that do not reflect at any $\alpha < \kappa$?

Capturing clubs

Suppose S is stationary subset of κ, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α.

Definition

We say that $\vec{\mathcal{F}}$ captures clubs iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is non-stationary;

For any ordinal $\alpha < \kappa$ of uncountable cofinality, denote by $CUB(\alpha)$ the club filter of subsets of α. The sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S^\kappa_{\omega_1} \rangle$ define by $\mathcal{F}_\alpha = CUB(\alpha)$, capture clubs.

Filter reflection

Suppose X and S are stationary subsets of κ, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α.

Definition

We say that X $\vec{\mathcal{F}}$-reflects to S iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary

Definition

We say that X $\mathfrak{f}$-reflects to S iff there exists a sequence of filters $\vec{\mathcal{F}}$ over a stationary subset S' of S such that X $\vec{\mathcal{F}}$-reflects to S'.

Some comments

- Suppose $X, S \subseteq \kappa$ are stationary sets such that every ordinal $\alpha \in S$ has uncountable cofinality and every stationary $Y \subseteq X$ reflects at stationary many $\beta \in S$. Define the sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ by $\mathcal{F}_\alpha = CUB(\alpha)$. Clearly X $\vec{\mathcal{F}}$-reflects to S.

- We call fake reflection the case when X $\mathfrak{f}$-reflects to S and for all $\alpha \in S$, $\mathcal{F}_\alpha \not\supseteq CUB(\alpha)$.

- Suppose $S \subseteq \kappa$ is stationary and $\{S_\beta \mid \beta < \kappa\}$ a partition of S. Define the sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ by: For all $\alpha \in S_\beta$ let $\mathcal{F}_\alpha$ be the filter generated by $\{\beta\}$ if $\beta < \alpha$, and $\{\alpha\}$ otherwise. Clearly for all $Y \subseteq X$, $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary.

Strong forms of filter reflection

Definition

We say that X strongly $\vec{\mathcal{F}}$-reflects to S iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha\}$ is stationary.

Definition

We say that X $\vec{\mathcal{F}}$-reflects with $\diamondsuit$ to S iff $\vec{\mathcal{F}}$ captures clubs and there exists a sequence $\langle Y_\alpha \mid \alpha \in S\rangle$ such that, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y_\alpha = Y \cap \alpha \;\&\; Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary.

Outline

Properties

Fact

For stationary subsets X and S of κ, $(1) \implies (2) \implies (3)$:

1. *X $\mathfrak{f}$-reflects with $\diamondsuit$ to S;*
2. *X strongly $\mathfrak{f}$-reflects to S;*
3. *X $\mathfrak{f}$-reflects to S.*

Fact (Monotonicity)

For stationary sets $Y \subseteq X \subseteq \kappa$ and $S \subseteq T \subseteq \kappa$:

1. *If X $\mathfrak{f}$-reflects to S, then Y $\mathfrak{f}$-reflects to T;*
2. *If X strongly $\mathfrak{f}$-reflects to S, then Y strongly $\mathfrak{f}$-reflects to T;*
3. *If X $\mathfrak{f}$-reflects with $\diamondsuit$ to S, then Y $\mathfrak{f}$-reflects with $\diamondsuit$ to T.*

Strong Filter Reflection

Proposition

Suppose X strongly $\mathfrak{f}$-reflects to S. If $\diamondsuit_X$ holds, then so does $\diamondsuit_S$.

Corollary

Assuming the consistency of a weakly compact cardinal, it is consistent that the two hold together:

- *Every stationary subset of $S^{\omega_2}_{\omega}$ reflects in $S^{\omega_2}_{\omega_1}$;*
- *There exists no stationary subset of $S^{\omega_2}_{\omega}$ that strongly $\mathfrak{f}$-reflects to $S^{\omega_2}_{\omega_1}$.*

Fake reflection and reductions

Lemma

If X $\mathfrak{f}$-reflects to S, then $=^\kappa_X \hookrightarrow_1 =^\kappa_S$.

Proof.

Suppose that $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ witnesses that X $\mathfrak{f}$-reflects to S. For every $\alpha \in S'$, define an equivalence relation $\sim_\alpha$ over κ^α by letting $\eta \sim_\alpha \xi$ iff there is $W \in \mathcal{F}_\alpha$ such that $W \cap X \subseteq \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\}$. As there are at most $|\kappa^\alpha|$ many equivalence classes and as $\kappa^{<\kappa} = \kappa$, we may attach to each equivalence class $[\eta]_{\sim_\alpha}$ a unique ordinal (a *code*) in κ, which we shall denote by $\ulcorner [\eta]_{\sim_\alpha} \urcorner$. Next, define a map $f : \kappa^\kappa \to \kappa^\kappa$ by letting for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$:

$$f(\eta)(\alpha) := \begin{cases} \ulcorner [\eta \restriction \alpha]_{\sim_\alpha} \urcorner, & \text{if } \alpha \in S'; \\ 0, & \text{otherwise.} \end{cases}$$

■

Fake reflection and reductions

Lemma

If X strongly $\mathfrak{f}$-reflects to S, then for all $\theta \in [2, \kappa]$, $=^\theta_X \hookrightarrow_1 =^\theta_S$.

Proof.

We may assume that $\theta \in [2, \kappa)$. Suppose $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ is a sequence witnessing that X strongly $\mathfrak{f}$-reflects to S. Define a map $f : \theta^\kappa \to \theta^\kappa$ as follows. For every $\alpha \in S'$ and $\eta \in \theta^\kappa$, if there exists $W \in \mathcal{F}_\alpha$ and $i < \theta$ such that $W \cap X \subseteq \{\beta < \alpha \mid \eta(\beta) = i\}$, then it is unique (since $\mathcal{F}_\alpha$ is a filter), and so we let $f(\eta)(\alpha) := i$. If there is no such i or if $\alpha \notin S'$, then we simply let $f(\eta)(\alpha) := 0$. ■

Fake reflection and reductions

Lemma

If X $\mathfrak{f}$-reflects with $\diamondsuit$ to S, then $\leqslant^X \hookrightarrow_1 \subseteq^S$.

Proof.

Let $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ and $\langle Y_\alpha \mid \alpha \in S' \rangle$ witness together that X $\mathfrak{f}$-reflects with $\diamondsuit$ to S. Let $S'' := \{\alpha \in S' \mid Y_\alpha \in \mathcal{F}_\alpha^+\}$. For each $\alpha \in S''$, let $\bar{\mathcal{F}}_\alpha$ be the filter over α generated by $\mathcal{F}_\alpha \cup \{Y_\alpha\}$.
Claim: There exists a sequence $\langle \eta_\alpha \mid \alpha \in S'' \rangle$ such that, for every stationary $Y \subseteq X$ and every $\eta \in \kappa^\kappa$, the set $\{\alpha \in S'' \mid \eta_\alpha = \eta \restriction \alpha \ \& \ Y \cap \alpha \in \bar{\mathcal{F}}_\alpha\}$ is stationary. ∎

Fake reflection and reductions

Lemma

If X $\mathfrak{f}$-reflects with $\diamondsuit$ to S, then $\leqslant^X \hookrightarrow_1 \subseteq^S$.

Proof.

Let $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ and $\langle Y_\alpha \mid \alpha \in S' \rangle$ witness together that X $\mathfrak{f}$-reflects with $\diamondsuit$ to S. Let $S'' := \{\alpha \in S' \mid Y_\alpha \in \mathcal{F}^+_\alpha\}$. For each $\alpha \in S''$, let $\bar{\mathcal{F}}_\alpha$ be the filter over α generated by $\mathcal{F}_\alpha \cup \{Y_\alpha\}$. Let $\langle \eta_\alpha \mid \alpha \in S'' \rangle$ be given by the preceding claim. For every $\alpha \in S''$, define a quasi-order $\preccurlyeq_\alpha$ over κ^α by letting $\eta \preccurlyeq_\alpha \xi$ iff there is $W \in \bar{\mathcal{F}}_\alpha$ such that $W \cap X \subseteq \{\beta < \alpha \mid \eta(\beta) \leqslant \xi(\beta)\}$. Define a map $f : \kappa^\kappa \to 2^\kappa$ by letting for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$:

$$f(\eta)(\alpha) := \begin{cases} 1, & \text{if } \alpha \in S'' \;\&\; \eta_\alpha \preccurlyeq_\alpha \eta \restriction \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

■

Not Σ_1^1-complete

Question (Aspero-Hyttinen-Kulikov-M, Question 4.3)

Is it consistent that κ is inaccessible and $=^2_S$ is not Σ_1^1-complete for some stationary $S \subseteq \kappa$?

Theorem

If κ is an inaccessible cardinal, then there exists a cofinality-preserving forcing extension in which (κ is inaccessible, and) for every stationary co-stationary $S \subseteq \kappa$, $=^2_S$ is not a Σ_1^1-complete equivalence relation.

Not reduction

Question (M, Question 4.16)

Is it consistent that

$$=_\mu \not\hookrightarrow_B =_\nu$$

holds for all infinite regular cardinals $\mu \neq \nu$ below κ?

Theorem

There is a cofinality-preserving forcing extension, in which, for all infinite regular cardinals $\mu \neq \nu$ below κ, $=_\mu \not\hookrightarrow_{BM} =_\nu$.

Question (Aspero-Hyttinen-Kulikov-M, Question 2.12)

Is it consistent that, for all infinite regular $\mu < \nu < \kappa$, the following hold?

$$=_\mu \hookrightarrow_B =^2_\nu \ \& \ =^2_\nu \not\hookrightarrow_B =_\mu .$$

Theorem

Suppose MM *holds. After forcing with* $\mathrm{Add}(\omega_2, \omega_3)$, MM *still holds, and so are all of the following:*

1. $=^{\omega_2}_{\omega} \hookrightarrow_1 =^2_{\omega_1}$;
2. *For every stationary* $X \subseteq S^{\omega_2}_{\omega_1}$, $=^2_X \not\hookrightarrow_{BM} =^{\omega_2}_{\omega}$;
3. *There are stationary subsets* $X \subseteq S^{\omega_2}_{\omega}$ *and* $Y \subseteq S^{\omega_2}_{\omega_1}$ *such that* $=^2_X \not\hookrightarrow_{BM} =^{\omega_2}_Y$;
4. *There is a stationary* $Y \subseteq S^{\omega_2}_{\omega_1}$ *such that* $=^2_{\omega_1} \not\hookrightarrow_{BM} =^{\omega_2}_Y$;
5. $=^{\omega_2}_{\omega} \hookrightarrow_1 =^2_{\omega_1}$ *and* $=^2_{\omega_1} \not\hookrightarrow_{BM} =^{\omega_2}_{\omega}$.

Over the limits

- Usual stationary reflection is a special case of filter reflection.
- For all regular cardinals $\gamma \leqslant \lambda < \kappa$, any $X \subseteq S^\kappa_\lambda$, X does not reflect at any $\alpha \in S^\kappa_\gamma$. S^κ_λ $\mathfrak{f}$-reflects to S^κ_γ is consistently true.
- If $\kappa = \lambda^+$ and $\Box_\lambda$ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that Y does not reflect at any $\alpha < \kappa$. Fake reflection is consistent with $\Box_\lambda$.
- Fake reflection does not require large cardinals. This is the case of L.

Outline

What happens in L

Suppose $V = L$. For $\kappa = \lambda^+$, it is known that for all stationary sets $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ that does not reflect at any $\alpha < \kappa$.

Question

What about fake reflection?
Suppose $V = L$. Does X $\mathfrak{f}$-reflects to κ, for all stationary $X \subseteq \kappa$?

A diamond reflection principle

For sets N and x, we say that N *sees* x iff N is a transitive model of ZF^- and $x \cup \{x\} \subseteq N$

Definition

For a stationary $S \subseteq \kappa$ and a positive integer n, $DI^*_S(\Pi^1_n)$ asserts the existence of a sequence $\vec{N} = \langle N_\alpha \mid \alpha \in S \rangle$ satisfying the following:

1. *for every $\alpha \in S$, N_α is a set of cardinality $< \kappa$ that sees α;*
2. *for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$;*
3. *for every Π^1_n-sentence ϕ valid in a structure $\langle \kappa, \in, (A_m)_{m \in \omega} \rangle$, there are stationarily many $\alpha \in S$ such that $|N_\alpha| = |\alpha|$ and*

$$N_\alpha \models \text{“}\phi \text{ is valid in } \langle \alpha, \in, (A_m \restriction \alpha)_{m \in \omega} \rangle\text{”}.$$

$DI^*_S(\Pi^1_1)$ and fake reflection

Lemma

Suppose $S \subseteq \kappa$ is stationary for which $\mathrm{DI}^*_S(\Pi^1_1)$ holds. Then for all stationary $X \subseteq \kappa$, X $\mathfrak{f}$-reflects to S.

Proof.

Idea: Let Φ be a Π^1_1-sentence such that for all α, $\langle \alpha, \in \rangle \models \Phi$ if and only if α is regular. Let $S' \subseteq S$ be the set of ordinals such that $N_\alpha \models$ "Φ *is valid in* $\langle \alpha, \in \rangle$". For all $\alpha \in S'$, define $\mathcal{F}_\alpha$ as the set of $D \in N_\alpha$ such that $N_\alpha \models$ "*D is a club*". ∎

Fake reflection in L

Theorem

Suppose $V = L$. For any stationary set $S \subseteq \kappa$, $DI^*_S(\Pi^1_2)$ holds.

Corollary

Suppose $V = L$. Then for every stationary set $S \subseteq \kappa$, κ $\mathfrak{f}$-reflects to S.

Remark

By monotonicity, suppose $V = L$, then for all stationary sets $X, S \subseteq \kappa$, X $\mathfrak{f}$-reflects to S.

In particular S $\mathfrak{f}$-reflects to S and $S^\kappa_{\omega_1}$ $\mathfrak{f}$-reflects to S^κ_ω.

The next step

Question

Can we force filter reflection?

Easy answer: Yes. Just force usual stationary reflection (collapse a weakly compact cardinal).

Question

Can we force fake reflection without using large cardinals?

Sakai's forcing

Definition

Let $\mathbb{S}$ be the poset of all pairs $(k, \mathcal{B})$ with the following properties:

1. *k is a function such that $dom(k) < \kappa$;*
2. *for each $\alpha \in dom(k), k(\alpha)$ is a transitive model of ZF^- of size $\leq \max\{\aleph_0, |\alpha|\}$, with $k \restriction \alpha \in k(\alpha)$;*
3. *$\mathcal{B}$ is a subset of $\mathcal{P}(\kappa)$ of size $\leq \mathrm{dom}(k)$;*

$(k', \mathcal{B}') \leqslant (k, \mathcal{B})$ in $\mathbb{S}$ if the following holds:

(i) *$k' \supseteq k$, and $\mathcal{B}' \supseteq \mathcal{B}$;*

(ii) *for any $B \in \mathcal{B}$ and any $\alpha \in dom(k') \setminus dom(k)$, $B \cap \alpha \in k'(\alpha)$.*

Fact

For every stationary $S \subseteq \kappa$, $V^{\mathbb{S}} \models DI^*_S(\Pi^1_n)$.

Conclusion

Corollary

For all stationary subsets X and S of κ, there exists a $<\kappa$-closed κ^+-cc forcing extension, in which X $\mathfrak{f}$-reflects to S.

Outline

The Failure

Question

Is the failure of filter reflection consistently true?

- Weakly compact: clearly the failure cannot be forced.
- Usual stationary reflection: force $\square_\lambda$.
- Fake reflection: forcing $\square_\lambda$ is not enough.

Question

What do we need to kill fake reflection?

$I[\kappa - X]$

Definition

Let $X \subseteq \kappa$. We define a collection $I[\kappa - X]$, as follows.
A set Y is in $I[\kappa - X]$ iff $Y \subseteq \kappa$ and there exists a sequence $\langle a_\beta \mid \beta < \kappa\rangle$ of elements of $[\kappa]^{<\kappa}$ along with a club $C \subseteq \kappa$ such that, for every $\delta \in Y \cap C$, there is a cofinal subset $A \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ such that

1. $\{A \cap \gamma \mid \gamma < \delta\} \subseteq \{a_\beta \mid \beta < \delta\}$, *and*
2. $\mathrm{acc}^+(A) \cap X = \emptyset$.

Shelah's *approachability ideal* $I[\kappa]$ is equal to $I[\kappa - \emptyset] \restriction \mathrm{Sing}$

$\mathrm{Add}(\kappa, 1)$

Theorem

Suppose X, S are stationary subsets of κ, with $S \in I[\kappa - X]$. For every $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$, $V^{\mathrm{Add}(\kappa,1)} \models X$ does not $\vec{\mathcal{F}}$-reflect to S.

Proof: Towards a contradiction, suppose that $\vec{\mathcal{F}}$ is a counterexample.

Let R denote the set of all pairs $(p, q) \in 2^{<\kappa} \times 2^{<\kappa}$ such that:

- $\mathrm{dom}(p) = \mathrm{dom}(q)$ is in $\mathrm{nacc}(\kappa)$;
- $\{\alpha \in \mathrm{dom}(p) \mid p(\alpha) = q(\alpha) = 1\}$ is disjoint from X;
- $\{\alpha \in \mathrm{dom}(q) \mid q(\alpha) = 1\}$ is a closed set of ordinals.

We let $\mathbb{R} := (R, \leq)$ where $(p', q') \leq (p, q)$ iff $p' \supseteq p$ and $q' \supseteq q$.

continuation of the proof

$\mathbb{R}$ is $<\kappa$-closed of size κ, $\mathbb{R}$ is forcing equivalent to $\mathrm{Add}(\kappa, 1)$.

Let $P := \{p \mid \exists q\ (p, q) \in R\}$. It is easy to see that $\mathbb{P} := (P, \supseteq)$ is $<\kappa$-closed, so that $\mathbb{P}$ is forcing equivalent to $\mathrm{Add}(\kappa, 1)$.

Let G be $\mathbb{R}$-generic over V. Let G_0 denote the projection of G to the first coordinate, so that G_0 is $\mathbb{P}$-generic over V.

In $V[G_0]$, let $Q := \{q \in 2^{<\kappa} \mid \exists p \in G_0\ (p, q) \in R\}$. Clearly, $\mathbb{Q} := (Q, \supseteq)$ is isomorphic to the quotient forcing $\mathbb{R}/G_0$.

It follows that, in $V[G]$, we may read a $\mathbb{Q}$-generic set G_1 over $V[G_0]$ such that, in particular, $V[G] = V[G_0][G_1]$.

Denote $\eta := \bigcup G_0$ and let $Y := \{\alpha \in X \mid \eta(\alpha) = 1\}$.

continuation of the proof

Recall: $\eta := \bigcup G_0$ and $Y := \{\alpha \in X \mid \eta(\alpha) = 1\}$. Define $T := \{\alpha \in S \setminus Y \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$.
We will prove the following claim:

1. In $V[G_0]$, Y and T are stationary.
2. In $V[G_0][G_1]$, T is stationary.
3. In $V[G_0][G_1]$, Y is nonstationary.

The last two claims contradict that $\vec{\mathcal{F}}$ capture clubs.

Killing fake reflection

Corollary

Suppose X, S are stationary subsets of κ, with $S \in I[\kappa - X]$. After forcing with $\mathrm{Add}(\kappa, \kappa^+)$, X does not $\mathfrak{f}$-reflect to S.

By doing a preliminary forcing to enlarge $I[\kappa - X]$ for all X, we obtain:

Corollary (Dense non-reflection)

There exists a cofinality-preserving forcing extension in which for all two stationary subsets X, S of κ, X does not $\mathfrak{f}$-reflect to S.

Killing fake reflection

Lemma

Suppose that κ *is strongly inaccessible or* $\kappa = \lambda^+$ *with* $\lambda^{<\lambda} = \lambda$*. For every stationary* $X, Y \subseteq \kappa$ *such that* $\mathrm{Tr}(X) \cap Y$ *is non-stationary,* $Y \in I[\kappa - X]$*.*

Corollary

If κ *is strongly inaccessible (e.g.,* κ *Laver-indestructible supercompact), then in the forcing extension by* $\mathrm{Add}(\kappa, \kappa^+)$*, for all two stationary subsets* X, S *of* κ*, the following are equivalent:*

1. X $\mathfrak{f}$*-reflects to* S*;*
2. *every stationary subset of* X *reflects in* S*.*

Thank you