Finding the main gap in the generalised descriptive set theory

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The spectrum problem

Let T be a countable theory over a countable language. Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T,\alpha)$?

- **Löwenheim-Skolem Theorem:**

$$\exists\alpha \geq \omega\ I(T,\alpha) \neq 0 \Rightarrow \forall\beta \geq \omega\ I(T,\beta) \neq 0.$$

- **Morley's categoricity:**

$$\exists\alpha > \omega\ I(T,\alpha) = 1 \Rightarrow \forall\beta > \omega\ I(T,\beta) = 1.$$

Shelah's Main Gap Theorem

Theorem (Shelah)

Let T be a countable theory.

- *If T is not superstable or (is superstable) deep or with the DOP or the OTOP, then for every uncountable α, $I(T,\alpha)=2^{\alpha}$.*
- *If T is shallow superstable without the DOP and without the OTOP, then for every $\alpha>0$, $I(T,\aleph_\alpha)<\beth_{\omega_1}(\mid\alpha\mid)$.*

Shelah's Main Gap Theorem

Theorem (Shelah)

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Non-classifiable theories

A theory T is non-classifiable if it is a countable complete theory that satisfies one of the following:

- T is unstable;
- T is stable unsuperstable;
- T is superstable with DOP;
- T is superstable with OTOP.

The topology

Let κ be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

The Generalised Baire spaces

The generalised Baire space is the space κ^κ endowed with the bounded topology.

The generalised Cantor space is the subspace 2^κ.

Coding structures

Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$.

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in \kappa^\kappa$ define the structure $\mathcal{A}_f$ with domain κ and for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

The isomorphism relation

Definition

Given T a first-order complete countable theory in a countable vocabulary, we say that $f, g \in \kappa^\kappa$ are $\cong_T$ equivalent if
$\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
or $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

Question

Question. What can we say about the division lines using the isomorphism relation?

κ-Borel sets

The κ-Borel space of κ^κ is the smallest set, which contains the basic open sets, and is closed under unions and intersections, both of length κ, and complements.

A κ-Borel set, is any element of the κ-Borel space.

A set X is a κ-Σ_1^1 if it is a projection of a closed set $C \subseteq (\kappa^\kappa)^2$.

A set X is a κ-Π_1^1 if it is the complement of a κ-Σ_1^1 set.

A set X is a κ-Δ_1^1 if it is a κ-Σ_1^1 and κ-Π_1^1 set.

Classifiable theories

Theorem (Friedman - Hyttinen - Kulikov)

If $\kappa > 2^\omega$ is a succesor cardinal, then T is classifiable and shallow if and only if $\cong_T$ is κ-Borel.

Theorem (Friedman - Hyttinen - Kulikov)

If T is classifiable not shallow, then $\cong_T$ is κ-Δ^1_1 and not κ-Borel.

κ-Borel rank

If A is a κ-Borel set, the smallest ordinal $1 \leq \alpha \leq \kappa^+$ such that $A \in \Sigma^0_\alpha(\kappa) \cup \Pi^0_\alpha(\kappa)$ is called the κ-Borel rank of A and denoted by $rk_B(A)$.

Theorem (Mangraviti - Motto Ros)

Let κ be such that $\kappa > 2^\omega$. If T is classifiable and shallow with depth α, then $rk_B(\cong_T) \leq 4\alpha$.

Non-Classifiable theories

Theorem (Friedman - Hyttinen - Kulikov)

- *If T is unstable, then $\cong_T$ is κ-Σ_1^1 and not κ-Δ_1^1.*
- *If T is superstable with OTOP, then $\cong_T$ is κ-Σ_1^1 and not κ-Δ_1^1.*
- *If T is superstable with DOP and $\kappa > \omega_1$, then $\cong_T$ is κ-Σ_1^1 and not κ-Δ_1^1.*

Non-classifiableable theories

Question. Is consistent that there is a stable unsuperstable theory for which $\cong_T$ is κ-Δ_1^1?

Reductions

Let E_1 and E_2 be equivalence relations on θ^κ, $\theta \in \{2, \kappa\}$. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f : \theta^\kappa \to \theta^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$. We write $E_1 \hookrightarrow_B^\theta E_2$.

We can define a partial order on the set of all first-order complete countable theories

$$T \leq^\theta T' \textit{ iff } \cong_T \hookrightarrow_B^\theta \cong_{T'}$$

Classifiable theories

Theorem (Mangraviti - Motto Ros)

Let $\kappa = \aleph_\gamma$ be such that $\kappa^{<\kappa} = \kappa$ and $\beth_{\omega_1}(|\gamma|) \leq \kappa$. Let T, T' be countable complete first-order theories, and suppose T is classifiable and shallow, while T' is not. Then

$$T \ \leq^\kappa \ T'$$

Consistency of the Main Gap

Theorem (Hyttinen - Kulikov - M.)

Let $H(\kappa)$ be the property: If T is classifiable and T' is not, then $T \leq^\kappa T'$ and $T' \not\leq^\kappa T$. Suppose that $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, $\lambda^{<\lambda} = \lambda$.

- *$\diamondsuit(E^\kappa_\lambda)$ implies $H(\kappa)$.*
- *There is a κ-closed κ^+-cc which forces $H(\kappa)$.*

Completeness

Theorem (Fernandes - M. - Rinot)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, *and* $\lambda^{<\lambda} = \lambda$. *Let* T *be a non-classifiable theory. There is a* κ*-closed* κ^+*-cc forcing which forces: If* T' *is a countable complete first-order theory, then* $T' \leq^\kappa T$.

A dichotomy

Theorem (Hyttinen - Kulikov - M.)

The following is consistent:
For all first-order countable theory T one of the following holds:

1. $\cong_T$ *is* κ-Δ_1^1
2. *If T' is a countable complete first-order theory, then* $T' \leq^\kappa T$.

Stable unsuperstable theories

Theorem (Hyttinen - Kulikov - M.)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, *and* $\lambda^{<\lambda} = \lambda$. *If* T *is classifiable and* T' *is stable unsuperstable, then* $T \leq^\kappa T'$.

The orthogonal chain property

A stable theory T has the orthogonal chain property (OCP), if there exists $\lambda_r(T)$-saturated models of T of power $\lambda_r(T)$, $\{\mathcal{A}_i\}_{i<\omega}$, and $a \notin \cup_{i<\omega}\mathcal{A}_i$, such that:

- For all $i<j$, $\mathcal{A}_i \subseteq \mathcal{A}_j$;
- $t(a, \cup_{i<\omega}\mathcal{A}_i)$ is not algebraic;

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- For all $i < j$, $\mathcal{A}_i \subseteq \mathcal{A}_j$;
- $t(a, \cup_{i<\omega}\mathcal{A}_i)$ is not algebraic;
- for all $j < \omega$, $t(a, \cup_{i<\omega}\mathcal{A}_i) \perp \mathcal{A}_j$.

The orthogonal chain property

Theorem (Hyttinen - M.)

If T has the OCP, then T is stable unsuperstable.

Theorem (Hyttinen - M.)

Let κ be an inaccessible cardinal. If T is a classifiable theory and T' is a theory with OCP, then $T \leq^\kappa T'$.

The strong dimensional order property

We say that a superstable theory T has the strong dimensional order property (S-DOP) if the following holds:

There are F^a_ω-saturated models $(M_i)_{i<3}$, $M_0 \subseteq M_1 \cap M_2$, such that $M_1 \downarrow_{M_0} M_2$, and for every M_3 F^a_ω-prime model over $M_1 \cup M_2$, there is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, such that it does not fork over $M_1 \cup M_2$.

The strong dimensional order property

Theorem (M.)

Let κ be an inaccessible cardinal. If T is a classifiable theory and T' is a superstable theory with S-DOP, then $T \leq^\kappa T'$.

Unsuperstable

Theorem (M.)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^\omega = \lambda$. If T is a countable complete classifiable theory, and T' is a countable complete unsuperstable theory, then $T \leq^\kappa T'$.

Forking sequences

Theorem (Feldman - M.)

Let κ be an inaccessible cardinal. Suppose T is such that there is an increasing sequence A_i, $i \leq \omega+\omega$, and $p \in S(A_{\omega+\omega})$ such that for all $i < \omega+\omega$, $p \restriction A_{i+1}$ forks over A_i.
Then for all classifiable theory T', $T' \leq^{\kappa} T$.

Thank you

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