The Borel reducibility Main Gap

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The spectrum fuction

Let T be a countable theory over a countable language. Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T,\alpha)$?

Categoricity

- **1915 - 1920:** Löwenheim-Skolem Theorem.
- **1929:** Gödel's completeness theorem.
- **1965:** Morley's categoricity theorem.
- **1960's:** Let T be a first-order countable theory over a countable language. For all $\aleph_0 < \lambda < \kappa$,

$$I(T, \lambda) \leq I(T, \kappa).$$

Shelah's Main Gap Theorem

Theorem (Shelah 1990)

Either, for every uncountable cardinal α, $I(T,\alpha) = 2^\alpha$; or $\forall \alpha > 0$, $I(T,\aleph_\alpha) < \beth_{\omega_1}(|\alpha|)$.

If T has less models than T', then T is less complex than T' and their complexity are not close.

Non-classifiable theories

A theory T is non-classifiable if it is a countable complete theory that satisfies one of the following:

- T is unstable;
- T is stable unsuperstable;
- T is superstable with DOP;
- T is superstable with OTOP.

Classifiable theories

Classifiable are divided into:

- shallow,
$$I(T,\aleph_\alpha) < \beth_{\omega_1}(|\alpha|);$$
- non-shallow,
$$I(T,\alpha) = 2^\alpha.$$

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Descriptive Set Theory

- **1989:** Friedman and Stanley introduced the Borel reducibility between classes of countable structures.
- **1993:** Mekler-Väänänen κ-separation theorem.
- **2014:** Friedman-Hyttinen-Kulikov developed GDST and a systematic comparison between the Main Gap dividing lines and the complexity given by Borel reducibility.

The bounded topology

Let κ be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

The Generalised Baire spaces

The generalised Baire space is the space κ^κ endowed with the bounded topology.

The generalised Cantor space is the subspace 2^κ.

Coding structures

Let $\omega \leq \mu \leq \kappa$ be a cardinal. Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$ and a bijection π_μ between $\mu^{<\omega}$ and μ.

Definition

For every $\eta \in \kappa^\kappa$ define the structure $\mathcal{A}_{\eta\restriction\mu}$ with domain μ as follows: For every tuple $(a_1, a_2, \ldots, a_n)$ in μ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_{\eta\restriction\mu}} \Leftrightarrow \eta(\pi_\mu(m, a_1, a_2, \ldots, a_n)) > 0.$$

The isomorphism relation

Definition

Let $\omega \leq \mu \leq \kappa$ be a cardinal and T a first-order theory in a relational countable language, we say that $\eta, \xi \in \kappa^\kappa$ are $\cong^\mu_T$ equivalent if one of the following holds:

- $\mathcal{A}_{\eta\restriction\mu} \models T, \mathcal{A}_{\xi\restriction\mu} \models T, \mathcal{A}_{\eta\restriction\mu} \cong \mathcal{A}_{\xi\restriction\mu}$
- $\mathcal{A}_{\eta\restriction\mu} \not\models T, \mathcal{A}_{\xi\restriction\mu} \not\models T$

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *reducible* to E_2, if there is a function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$. We write $E_1 \hookrightarrow_r E_2$.

We can use continuous functions to define a partial order on the set of all first-order complete countable theories

$$T \leq^\kappa T' \textit{ iff } \cong_T \hookrightarrow_C \cong_{T'}$$

Question

Question: What can we say about the Borel-reducibility between different dividing lines?

Conjecture: If T is classifiable and T' is non-classifiable, then $T \leq^\kappa T'$ and $T' \not\leq^\kappa T$.

Classifiable and shallow

Theorem (Mangraviti - Motto Ros 2020)

Let $\kappa = \aleph_\gamma$ be such that $\kappa^{<\kappa} = \kappa$ and $\beth_{\omega_1}(|\gamma|) \leq \kappa$. Let T, T' be countable complete first-order theories, and suppose T is classifiable and shallow, while T' is not. Then

$$\cong_T \hookrightarrow_B \cong_{T'}$$

Fact (Mangraviti-Motto Ros)

Let E_1 be a Borel equivalence relation with $\gamma \leq \kappa$ equivalence classes and E_2 be an equivalence relation with θ equivalence classes. If $\gamma \leq \theta$, then $E_1 \hookrightarrow_B E_2$.

Counting α-classes relation

Let $\alpha < \kappa$ be an ordinal and $0 < \varrho \leq \kappa$. $\eta\ \alpha_\varrho\ \xi$ if and only if one of the following holds:

- ϱ is finite:
  - $\eta(\alpha) = \xi(\alpha) < \varrho - 1$;
  - $\eta(\alpha), \xi(\alpha) \geq \varrho - 1$.
- ϱ is infinite:
  - $\eta(\alpha) = \xi(\alpha) < \varrho$;
  - $\eta(\alpha), \xi(\alpha) \geq \varrho$.

Gap: Shallow and Non-shallow

Theorem (M. 2023)

Suppose $\aleph_\mu = \kappa = \lambda^+ = 2^\lambda$ *is such that* $\beth_{\omega_1}(|\mu|) \leq \kappa$. *Let* T_0 *and* T_1 *be countable complete classifiable shallow theories such that* $1 < I(\kappa, T_0) < I(\kappa, T_1) = \varrho$, T_2 *be a countable complete theory not classifiable shallow. Then*

$$\cong_{T_0} \hookrightarrow_B 0_\varrho \hookrightarrow_L \cong_{T_1} \hookrightarrow_B 0_\kappa \hookrightarrow_L \cong_{T_2}$$

and

$$\cong_{T_2} \not\hookrightarrow_r 0_\kappa \not\hookrightarrow_r \cong_{T_1} \not\hookrightarrow_C 0_\varrho \not\hookrightarrow_r \cong_{T_0} .$$

Consistency

Theorem (Hyttinen - Kulikov - M. 2017)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, and $\lambda^{<\lambda} = \lambda$. There is a κ-closed κ^+-cc forcing which forces: If T is classifiable and T' is non-classifiable, then $T \leq^\kappa T'$ and $T' \not\leq^\kappa T$.

Theorem (Hyttinen - Kulikov - M. 2017)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, and $\lambda^\omega = \lambda$. If T is classifiable and T' is stable unsuperstable, then $T \leq^\kappa T'$ and $T' \not\leq^\kappa T$.

Borel-reducibility Main Gap

Theorem (M. 2023)

Let $\mathfrak{c} = 2^\omega$. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $2^\mathfrak{c} \leq \lambda = \lambda^{\omega_1}$. If T is a classifiable theory, and T' is a non-classifiable theory, then $T \ \leq^\kappa \ T'$ and $T' \ \not\leq^\kappa \ T$.

Equivalence modulo γ cofinality

Definition

We define the equivalence relation $=^2_\gamma \subseteq 2^\kappa \times 2^\kappa$, as follows: let $S = \{\alpha < \kappa \mid cf(\alpha) = \gamma\}$,

$$\eta =^2_\gamma \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

$\cong_T \hookrightarrow_C =^2_\mu$, $\kappa = \lambda^+$

Theory	$\lambda = \lambda^\gamma$	$\diamondsuit_\lambda$	$\mathrm{DI}^*_{S^\kappa_\gamma}(\Pi^1_1)$
Classifiable	$\omega \leq \mu \leq \gamma$	$\mu = \lambda$	$\mu = \gamma$
Non-classifiable	Indep	Indep	$\mu = \gamma$

$=^2_\mu \hookrightarrow_C \cong_T,\ \kappa = \lambda^+$

Theory	$\lambda = \lambda^\gamma$	$2^{\mathfrak{c}} \leq \lambda = \lambda^\gamma$	$2^{\mathfrak{c}} \leq \lambda = \lambda^{<\lambda}$ & $\diamondsuit_\lambda$
Stable Unsuper-stable	$\mu = \omega$	$\mu = \omega$	$\mu = \omega$
Unstable	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \lambda$
Superstable with OTOP	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \lambda$
Superstable with DOP	?	$\omega_1 \leq \mu \leq \gamma$	$\omega_1 \leq \mu \leq \lambda$

A bigger Gap

Theorem (M. 2023)

Suppose κ is inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{\omega_1}$. There exists a cofinality-preserving forcing extension in which the following holds:
If T_1 is classifiable and T_2 is not. Then there is a regular cardinal $\gamma < \kappa$ such that, if $X, Y \subseteq S^\kappa_\gamma$ are stationary and disjoint, then $=^2_X$ and $=^2_Y$ are strictly in between $\cong_{T_1}$ and $\cong_{T_2}$.

Main Gap Dichotomy

Theorem (M. 2023)

Let κ be inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^\mathfrak{c} \leq \lambda = \lambda^{<\omega_1}$. There exists a $<\kappa$-closed κ^+-cc forcing extension in which for any countable first-order theory in a countable vocabulary (not necessarily complete), T, one of the following holds:

- $\cong_T$ *is* $\Delta_1^1(\kappa)$*;*
- $\cong_T$ *is* $\Sigma_1^1(\kappa)$*-complete.*

Non-classifiable theories

Lemma (M. 2023)

Let κ be strongly inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. For all cardinals $\aleph_0 < \mu < \delta < \kappa$, if T is a non-classifiable theory then

$$\cong^\mu_T \hookrightarrow_C \cong^\delta_T \hookrightarrow_C \ id \ \hookrightarrow_C \cong_T .$$

Classifiable non-shallow

Lemma (M. 2023)

Suppose $\kappa=\lambda^+=2^\lambda$. *The following reduction is strict. Let* $2^{\mathfrak{c}}\leq\lambda=\lambda^{<\omega_1}$. *If* T_1 *is a classifiable non-shallow theory and* T_2 *is a non-classifiable theory, then*

$$\cong^{\lambda}_{T_2}\hookrightarrow_C\cong_{T_1}\hookrightarrow_C\cong_{T_2}.$$

Classifiable shallow

Lemma (M. 2023)

Suppose $\kappa = \lambda^+ = 2^\lambda$. *The following reductions are strict. Let* $\kappa = \aleph_\gamma$ *be such that* $\beth_{\omega_1}(|\gamma|) \leq \kappa$. *Suppose* T_1 *is a classifiable shallow theory,* T_2 *a classifiable non-shallow theory, and* T_3 *non-classifiable theory. Then*

$$\cong_{T_1} \hookrightarrow_B \cong^\lambda_{T_3} \hookrightarrow_C \cong_{T_2} .$$

Detailed

Theorem (M. 2023)

Let $\mathfrak{c} = 2^\omega$. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{\omega_1}$. If T is a classifiable theory, and T' is a non-classifiable theory, then there is $\gamma < \kappa$ such that

$$\cong_T \hookrightarrow_C =^2_\gamma \hookrightarrow_C \cong_{T'} \text{ and } =^2_\gamma \not\hookrightarrow_B \cong_T .$$

Classifiable theories

Theorem (Hyttinen - Kulikov - M. 2017)

Assume T is a classifiable theory and let $S=\{\alpha<\kappa \mid cf(\alpha)=\gamma\}$. If $\diamondsuit_S$ holds, then $\cong_T \hookrightarrow_C =^2_\gamma$.

Theorem (Friedman - Hyttinen - Kulikov 2014)

If T is a classifiable theory and $\gamma<\kappa$ is regular, then $=^2_\gamma \not\hookrightarrow_B \cong_T$.

Blue print of the proof

- Construct the reductions.
- Construct Ehrenfeucht-Mostowski models, such that

$$f \ =^2_\gamma \ g \text{ iff } \mathcal{M}^f \cong \mathcal{M}^g.$$

- Construct ordered trees, such that

$$f \ =^2_\gamma \ g \ \Leftrightarrow A_f \cong A_g.$$

κ^+, $(\gamma+2)$-tree*

Let $\gamma<\kappa$ be a regular cardinal. A κ^+, $(\gamma+2)$-tree* t is a tree with the following properties:

- t has a unique root.
- Every element of t has less than κ^+ immediate successors.
- All the branches of t have order type γ or $\gamma+1$.
- Every chain of length less than γ has a unique limit.

Isomorphism of κ^+, $(\gamma+2)$-tree*

Lemma (Hyttinen - Kulikov - M.)

Suppose $\gamma<\kappa$ is such that for all $\epsilon<\kappa$, $\epsilon^\gamma<\kappa$. For every $f,g\in 2^\kappa$ there are κ^+, $(\gamma+2)$-trees J_f and J_g such that*

$$f =^2_\gamma g \Leftrightarrow J_f \cong_{ct} J_g$$

where $\cong_{ct}$ is the isomorphism of κ^+, $(\gamma+2)$-tree.*

Ordered trees

Definition

Let $\gamma < \kappa$ be a regular cardinal and I a linear order. $(A, \prec, <)$ is an ordered tree if the following holds:

- $(A, \prec)$ is a κ^+, $(\gamma + 2)$-tree*.
- for all $x \in A$, $(succ(x), <)$ is isomorphic to I.

κ-colorable

Definition

Let I be a linear order of size κ. We say that I is *κ-colorable* if there is a function $F : I \to \kappa$ such that for all $B \subseteq I$, $|B| < \kappa$, $b \in I \backslash B$, and $p = tp_{bs}(b, B, I)$ such that the following hold: For all $\alpha \in \kappa$,

$$|\{a \in I \mid a \models p \;\&\; F(a) = \alpha\}| = \kappa.$$

Isomorphism of ordered trees

Theorem (M. 2023)

Suppose $\gamma < \kappa$ is such that for all $\epsilon < \kappa$, $\epsilon^\gamma < \kappa$, and there is a κ-colorable linear order I. For all $f \in 2^\kappa$ there is an ordered tree A_f such that for all $f, g \in 2^\kappa$,

$$f \ =^2_\gamma \ g \Leftrightarrow A_f \cong A_g.$$

The models

Example of DOP.

Suppose T is superstable with DOP in a countable relational vocabulary τ. Let τ^1 be a Skolemization of τ, and T^1 be a complete theory in τ^1 extending T and with Skolem-functions in τ. Then for every $f \in 2^\kappa$ we want a model $\mathcal{M}_1^f \models T^1$ with the following properties.

The models

1. There is a map $\mathcal{H} : A_f \to (dom\ \mathcal{M}_1^f)^n$ for some $n < \omega$, $\eta \mapsto a_\eta$, such that $\mathcal{M}_1^f$ is the Skolem hull of $\{a_\eta \mid \eta \in A_f\}$. Let us denote $\{a_\eta \mid \eta \in A_f\}$ by $Sk(\mathcal{M}_1^f)$.
2. $\mathcal{M}^f = \mathcal{M}_1^f \restriction \tau$ is a model of T.
3. $Sk(\mathcal{M}_1^f)$ is indiscernible in $\mathcal{M}_1^f$ relative to $L_{\omega_1\omega_1}$, i.e. if $tp_{at}(\bar{s}, \emptyset, A_f) = tp_{at}(\bar{s'}, \emptyset, A_f)$, then $tp_\Delta(\bar{a}_{\bar{s}}, \emptyset, \mathcal{M}_1^f) = tp_\Delta(\bar{a}_{\bar{s'}}, \emptyset, \mathcal{M}_1^f)$, where $\Delta = L_{\omega_1\omega_1}$.
4. There is a formula $\varphi \in L_{\omega_1\omega_1}(\tau)$ such that for all $\eta, \nu \in A_f$ and $m < \gamma$, if $A_f \models P_m(\eta) \wedge P_\gamma(\nu)$, then $\mathcal{M}^f \models \varphi(a_\nu, a_\eta)$ if and only if $A_f \models \eta \prec \nu$.

Coding trees

For every $f \in 2^\kappa$ let us define the order $K^D(f)$ by:

I. $dom\ K^D(f) = (dom\ A_f \times \{0\}) \cup (dom\ A_f \times \{1\})$.

II. For all $\eta \in A_f$, $(\eta, 0) <_{K^D(f)} (\eta, 1)$.

III. If $\eta, \xi \in A_f$, then $\eta \prec \xi$ if and only if

$$(\eta, 0) <_{K^D(f)} (\xi, 0) <_{K^D(f)} (\xi, 1) <_{K^D(f)} (\eta, 1).$$

IV. If $\eta, \xi \in A_f$, then $\eta < \xi$ if and only if $(\eta, 1) <_{K^D(f)} (\xi, 0)$.

ε-dense

Definition

Let I be a linear order of size κ and ε a regular cardinal smaller than κ. We say that I is *ε-dense* if the following holds.

If $A, B \subseteq I$ are subsets of size less than ε such that for all $a \in A$ and $b \in B$, $a < b$, then there is $c \in I$, such that for all $a \in A$ and $b \in B$, $a < c < b$.

The isomorphism theorem

Theorem (M. 2023)

Suppose T is a non-classifiable first order theory in a countable relational vocabulary τ. If I is (κ, ε)-nice and $(<\kappa)$-stable, then for all $f, g \in 2^\kappa$

$$f \ =^2_\gamma \ g \text{ iff } \mathcal{M}^f \cong \mathcal{M}^g.$$

Blue print of the proof

- Construct an ε-dense, (κ,ε)-nice, $(<\kappa)$-stable, and κ-colorable linear order.
- Construct ordered trees from the linear order.
- Construct skeletons from ordered trees, to construct Ehrenfeucht-Mostowski models.
- Prove the isomorphism theorem.
- Construct the reductions.

Existence

Let $\theta < \kappa$ be the smallest cardinal such that there is a ε-dense model of *DLO* of size θ.

Theorem (M. 2023)

Suppose κ is inaccessible, or $\kappa = \lambda^+$, $2^\theta \leq \lambda = \lambda^{<\varepsilon}$. There is a ε-dense, (κ, ε)-nice, $(< \kappa)$-stable, and κ-colorable linear order.

Construction

Let $\mathcal{Q}$ be a model of DLO of size $\theta < \kappa$, that is ε-dense.

Definition

Let $\kappa \times \mathcal{Q}$ be ordered by the lexicographic order, $\mathcal{I}^0$ be the set of functions $f : \varepsilon \to \kappa \times \mathcal{Q}$ such that $f(\alpha) = (f_1(\alpha), f_2(\alpha))$, for which $|\{\alpha \in \varepsilon \mid f_1(\alpha) \neq 0\}|$ is smaller than ε.
If $f, g \in \mathcal{I}^0$, then $f < g$ if and only if $f(\alpha) < g(\alpha)$, where α is the least number such that $f(\alpha) \neq g(\alpha)$.

Construction

Let us fix $\tau \in \mathcal{Q}$. Let I be the set of functions $f : \varepsilon \to (\{0\} \times \mathcal{I}^0) \cup (\kappa \times \mathcal{Q})$ such that the following hold:

- $f \restriction \{0\} : \{0\} \to \{0\} \times \mathcal{I}^0$.
- $f \restriction \varepsilon \backslash \{0\} : \varepsilon \backslash \{0\} \to \kappa \times \mathcal{Q}$.
- There is $\alpha < \varepsilon$ ordinal such that $\forall \beta > \alpha$, $f(\beta) = (0, \tau)$. We say that the least α with such property is the *depth* of f and we denote it by $dp(f)$;
- There are functions $f_1 : \varepsilon \to \kappa$ and $f_2 : \varepsilon \to \mathcal{I}^0 \cup \mathcal{Q}$ such that $f(\beta) = (f_1(\beta), f_2(\beta))$ and $f_1 \restriction dp(f) + 1$ is strictly increasing.

Construction

We say that $f < g$ if and only if one of the following holds:

- $f(0) \neq g(0)$ and $f_2(0) < g_2(0)$;
- let $\alpha = dp(g)$, $\forall \beta \leq \alpha$, $f(\beta) = g(\beta)$ and $f_1(\alpha + 1) \neq 0$;
- exists $\alpha > 0$ such that $\forall \beta < \alpha$, $f(\beta) = g(\beta)$, and $f_1(\alpha), g_1(\alpha) \neq 0$ and $g(\alpha) > f(\alpha)$.

Thank you

Article at: https://arxiv.org/abs/2308.07510