Consistency of Filter Reflection

Miguel Moreno
University of Vienna
Vilho, Yrjö and Kalle Väisälä Foundation of the Finnish Academy of Science and Letters

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Outline

Stationary reflection

Let α be a not ω cofinal ordinal. A set $C \subseteq \alpha$ is a club if it is closed and unbounded. A set $S \subseteq \alpha$ is stationary if for all club $C \subset \alpha$, $C \cap S \neq \emptyset$.

Definition

Let κ be a regular uncountable cardinal $\alpha \in \kappa$ be a not ω-cofinal ordinal, and a stationary $S \subseteq \kappa$, we say that S reflects at α if $S \cap \alpha$ is stationary in α

If κ is a weakly compact cardinal, every stationary subset of κ reflects at a regular cardinal $\alpha < \kappa$.

Generalised descriptive set theory

Suppose κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

The generalised Baire space is the space κ^κ endowed with the bounded topology, for every $\eta \in \kappa^{<\kappa}$ the following set

$$N_\eta = \{\xi \in \kappa^\kappa \mid \eta \subseteq \xi\}$$

is a basic open set.

Equivalence modulo nonstationary

Definition

For every stationary set $S \subseteq \kappa$ and $\theta \in [2, \kappa]$, the equivalence relation $=^\theta_S$ over the subspace θ^κ is defined via

$$\eta =^\theta_S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\} \text{ is non-stationary.}$$

Definition

The quasi-order $\leq^S$ over κ^κ is defined via

$$\eta \leq^S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \text{ is non-stationary.}$$

The quasi-order $\subseteq^S$ over 2^κ is nothing but $\leq^S \cap (2^\kappa \times 2^\kappa)$.

Reductions

For $i < 2$, let X_i be some space from the collection $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$. Let R_0 and R_1 be binary relations over X_0 and X_1, respectively.

Definition

A function $f : X_0 \to X_1$ is said to be a reduction of R_0 to R_1 iff, for all $\eta, \xi \in X_0$,

$$\eta \; R_0 \; \xi \text{ iff } f(\eta) \; R_1 \; f(\xi).$$

The existence of a function f satisfying this is denoted by $R_0 \hookrightarrow R_1$.

Lipschitz reductions

For $i < 2$, let X_i be some space from the collection $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$. Let R_0 and R_1 be binary relations over X_0 and X_1, respectively.

For all $\eta, \xi \in \kappa^\kappa$, denote

$$\Delta(\eta, \xi) := \min(\{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cup \{\kappa\}).$$

A reduction f of R_0 to R_1 is said to be 1-Lipschitz iff for all $\eta, \xi \in X_0$,

$$\Delta(\eta, \xi) \leq \Delta(f(\eta), f(\xi)).$$

The existence of a 1-Lipschitz reduction f is denoted by $R_0 \hookrightarrow_1 R_1$. We likewise define $R_0 \hookrightarrow_c R_1$, $R_0 \hookrightarrow_B R_1$ and $R_0 \hookrightarrow_{BM} R_1$ once we replace 1-Lipschitz by a continuous, Borel, or Baire measurable map, respectively.

Comparing $=^{\kappa}_{S}$ and $=^{2}_{S}$

Fact (Asperó-Hyttinen-Kulikov-M)

If every stationary subset of X reflects at stationary many $\alpha \in Y$, then $=^{\kappa}_{X} \hookrightarrow_{c} =^{\kappa}_{Y}$.

Fact (Friedman-Hyttinen-Kulikov)

Suppose $V = L$, and $X \subseteq \kappa$ and $Y \subseteq reg(\kappa)$ are stationary. If every stationary subset of X reflects at stationary many $\alpha \in Y$, then $=^{2}_{X} \hookrightarrow_{c} =^{2}_{Y}$.

Limitations

Let λ be a regular cardinal and denote by S_λ the set $\{\alpha < \kappa \mid cf(\alpha) = \lambda\}$.

- For all regular cardinals $\gamma < \lambda < \kappa$, any $X \subseteq S_\lambda$, X does not reflect at any $\alpha \in S_\gamma$.
- If $\kappa = \lambda^+$ and $\square_\lambda$ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that Y does not reflect at any $\alpha < \kappa$. This happens in L.
- For all regular cardinal $\lambda < \kappa$, any $X \subseteq S_\lambda$, X does not reflect at any $\alpha \in S_\lambda$.
- *Full* stationary reflection usually requires large cardinals.

Outline

The case of L

Let us denote by $=^{\theta}_{\lambda}$ the relation $=^{\theta}_{S}$ when $S = S_{\lambda}$.

Fact (Hyttinen-Kulikov-M)

Suppose $V = L$. Let λ be a regular cardinal below κ. Then for all stationary $X \subseteq \kappa$, $=^{\kappa}_{X} \hookrightarrow_c =^{2}_{\lambda}$.

Question

How is this possible if there are sets in L that do not reflect at any $\alpha < \kappa$?

Capturing clubs

Suppose S is stationary subset of κ, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α.

Definition

We say that $\vec{\mathcal{F}}$ captures clubs iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is non-stationary;

For any $\alpha < \kappa$ not ω-cofinal, denote by $CUB(\alpha)$ the club filter of subsets of α. The sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S_{\omega_1} \rangle$ define by $\mathcal{F}_\alpha = CUB(\alpha)$, capture clubs.

Filter reflection

Suppose X and S are stationary subsets of κ, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α.

Definition

We say that X $\vec{\mathcal{F}}$-reflects to S iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary

Definition

We say that X $\mathfrak{f}$-reflects to S iff there exists a sequence of filters $\vec{\mathcal{F}}$ over a stationary subset S' of S such that X $\vec{\mathcal{F}}$-reflects to S'.

Some comments

- Suppose $X, S \subseteq \kappa$ are stationary sets such that every $\alpha \in S$ is not ω-cofinal and every stationary $Y \subseteq X$ reflects at stationary many $\beta \in S$. Define the sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ by $\mathcal{F}_\alpha = CUB(\alpha)$. Clearly X $\vec{\mathcal{F}}$-reflects to S.

- We call fake reflection the case when X $\mathfrak{f}$-reflects to S and for all $\alpha \in S$, $\mathcal{F}_\alpha \not\supseteq CUB(\alpha)$.

- Suppose $S \subseteq \kappa$ is stationary and $\{S_\beta \mid \beta < \kappa\}$ a partition of S. Define the sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ by: For all $\alpha \in S_\beta$ let $\mathcal{F}_\alpha$ be the filter generated by $\{\beta\}$ if $\beta < \alpha$, and $\{\alpha\}$ otherwise. Clearly for all $Y \subseteq X$, $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary.

Strong forms of filter reflection

Definition

We say that X strongly $\vec{\mathcal{F}}$-reflects to S iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha\}$ is stationary.

Definition

We say that X $\vec{\mathcal{F}}$-reflects with $\diamondsuit$ to S iff $\vec{\mathcal{F}}$ captures clubs and there exists a sequence $\langle Y_\alpha \mid \alpha \in S \rangle$ such that, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y_\alpha = Y \cap \alpha \ \& \ Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ is stationary.

Outline

Properties

Fact (Monotonicity)

For stationary sets $Y \subseteq X \subseteq \kappa$ and $S \subseteq T \subseteq \kappa$:

1. *If X $\mathfrak{f}$-reflects to S, then Y $\mathfrak{f}$-reflects to T;*
2. *If X strongly $\mathfrak{f}$-reflects to S, then Y strongly $\mathfrak{f}$-reflects to T;*
3. *If X $\mathfrak{f}$-reflects with $\diamondsuit$ to S, then Y $\mathfrak{f}$-reflects with $\diamondsuit$ to T.*

Proposition

Suppose X strongly $\mathfrak{f}$-reflects to S. If $\diamondsuit_X$ holds, then so does $\diamondsuit_S$.

Fake reflection and reductions

Lemma

If X $\mathfrak{f}$-reflects to S, then $=^\kappa_X \hookrightarrow_1 =^\kappa_S$.

Proof.

Suppose that $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ witnesses that X $\mathfrak{f}$-reflects to S. For every $\alpha \in S'$, define an equivalence relation $\sim_\alpha$ over κ^α by letting $\eta \sim_\alpha \xi$ iff there is $W \in \mathcal{F}_\alpha$ such that $W \cap X \subseteq \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\}$. As there are at most $|\kappa^\alpha|$ many equivalence classes and as $\kappa^{<\kappa} = \kappa$, we may attach to each equivalence class $[\eta]_{\sim_\alpha}$ a unique ordinal (a *code*) in κ, which we shall denote by $\ulcorner [\eta]_{\sim_\alpha} \urcorner$. Next, define a map $f : \kappa^\kappa \to \kappa^\kappa$ by letting for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$:

$$f(\eta)(\alpha) := \begin{cases} \ulcorner [\eta \restriction \alpha]_{\sim_\alpha} \urcorner, & \text{if } \alpha \in S'; \\ 0, & \text{otherwise.} \end{cases}$$

■

What happens in L

Suppose $V = L$. For $\kappa = \lambda^+$, it is known that for all stationary sets $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ that does not reflect at any $\alpha < \kappa$.

Question

What about fake reflection?
Suppose $V = L$. Does X $\mathfrak{f}$-reflects to κ, for all stationary $X \subseteq \kappa$?

A diamond reflection principle

For sets N and x, we say that N *sees* x iff N is transitive, p.r.-closed, and $x \cup \{x\} \subseteq N$

Definition

For a stationary $S \subseteq \kappa$ and a positive integer n, $DI^*_S(\Pi^1_n)$ asserts the existence of a sequence $\vec{N} = \langle N_\alpha \mid \alpha \in S \rangle$ satisfying the following:

1. *for every $\alpha \in S$, N_α is a set of cardinality $< \kappa$ that sees α;*
2. *for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$;*
3. *for every Π^1_n-sentence ϕ valid in a structure $\langle \kappa, \in, (A_m)_{m \in \omega} \rangle$, there are stationarily many $\alpha \in S$ such that $|N_\alpha| = |\alpha|$ and*

$$N_\alpha \models \text{“}\phi \text{ is valid in } \langle \alpha, \in, (A_m \restriction \alpha)_{m \in \omega} \rangle\text{”}.$$

$DI^*_S(\Pi^1_1)$ and fake reflection

Lemma

Suppose $S \subseteq \kappa$ is stationary for which $\mathrm{DI}^*_S(\Pi^1_1)$ holds. Then for all stationary $X \subseteq \kappa$, X $\mathfrak{f}$-reflects to S.

Proof.

Idea: Let Φ be a Π^1_1-sentence such that for all α, $\langle \alpha, \in \rangle \models \Phi$ if and only if α is regular. Let $S' \subseteq S$ be the set of ordinals such that $N_\alpha \models$ "Φ *is valid in* $\langle \alpha, \in \rangle$". For all $\alpha \in S'$, define $\mathcal{F}_\alpha$ as the set of $D \in N_\alpha$ such that $N_\alpha \models$ "*D is a club*". ∎

Fake reflection in L

Theorem

Suppose $V = L$. For all $n < \omega$ and any stationary set $S \subseteq \kappa$, $DI^*_S(\Pi^1_n)$ holds.

Corollary

Suppose $V = L$. Then for every stationary set $S \subseteq \kappa$, κ $\mathfrak{f}$-reflects to S.

Remark

By monotonicity, suppose $V = L$, then for all stationary sets $X, S \subseteq \kappa$, X $\mathfrak{f}$-reflects to S.

In particular S $\mathfrak{f}$-reflects to S and S_{ω_1} $\mathfrak{f}$-reflects to S_ω.

Over the limits

- Full stationary reflection is a special case of filter reflection.
- For all regular cardinals $\gamma < \lambda < \kappa$, any $X \subseteq S_\lambda$, X does not reflect at any $\alpha \in S_\gamma$. S_λ $\mathfrak{f}$-reflects to S_γ is consistently true.
- If $\kappa = \lambda^+$ and $\square_\lambda$ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that Y does not reflect at any $\alpha < \kappa$. Fake reflection is consistent with $\square_\lambda$.
- For all regular cardinal $\lambda < \kappa$, any $X \subseteq S_\lambda$, X does not reflect at any $\alpha \in S_\lambda$. S_λ $\mathfrak{f}$-reflects to S_λ is consistently true.
- Fake reflection does not requires large cardinals. This is the case of L.

Outline

The next step

Question

Can we force filter reflection?

Easy answer: Yes. Just force full reflection (collapse a weakly compact cardinal).

Question

Can we force fake reflection?

Sakai's $\diamondsuit^{++}$

Definition

For a stationary $S \subseteq \kappa$, $\diamondsuit_S^{++}$ asserts the existence of a sequence $\langle K_\alpha \mid \alpha \in S\rangle$ satisfying the following:

1. *for every infinite $\alpha \in S$, K_α is a set of size $|\alpha|$;*
2. *for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $C \cap \alpha, X \cap \alpha \in K_\alpha$;*
3. *the following set is stationary in $[H_{\kappa^+}]^{<\kappa}$:*

$$\{M \in [H_{\kappa^+}]^{<\kappa} \mid M \cap \kappa \in S \ \& \ \mathrm{clps}(M, \in) = (K_{M\cap\kappa}, \in)\}.$$

$\diamondsuit^{++}$ and $DI^*_S\Pi^1_n$

Lemma

For every stationary $S \subseteq \kappa$, $\diamondsuit^{++}_S$ implies $DI^*_S(\Pi^1_2)$.

Proof (sketch): Suppose $\langle K_\alpha \mid \alpha \in S\rangle$ is a $\diamondsuit^{++}_S$-sequence. Define a sequence $\vec{N} = \langle N_\alpha \mid \alpha \in S\rangle$ by letting N_α be the p.r.-closure of $K_\alpha \cup (\alpha + 1)$. Let $\phi = \forall X \exists Y \varphi$ be a Π^1_2-sentence and $(A_m)_{m\in\omega}$ be such that $\langle \kappa, \in, (A_m)_{m\in\omega}\rangle \models \phi$. Given an arbitrary club $C \subseteq \kappa$, we consider the following set

$$\mathcal{C} := \{M \prec H_{\kappa^+} \mid M \cap \kappa \in C \;\&\; (A_m)_{m\in\omega} \in M\}.$$

$\mathcal{C}$ is a club in $[H_{\kappa^+}]^{<\kappa}$.

$\diamondsuit^{++}$ and $DI^*_S\Pi^1_n$

Lemma

For every stationary $S \subseteq \kappa$, $\diamondsuit^{++}_S$ implies $DI^*_S(\Pi^1_2)$.

Proof continuation (sketch): By $\diamondsuit^{++}_S$ the set

$$\mathcal{C} \cap \{M \in [H_{\kappa^+}]^{<\kappa} \mid M \cap \kappa \in S \ \&\ \text{clps}(M, \in) = (K_{M\cap\kappa}, \in)\}$$

is stationary, pick M in this set. Since $\langle \kappa, \in, (A_m)_{m\in\omega}\rangle \models \phi$, by definition

$$H_{\kappa^+} \models \text{“}\forall X \subseteq \kappa^{m(\mathbb{X})}\ \exists Y \subseteq \kappa^{m(\mathbb{Y})}\ \langle \kappa, \in, (A_m)_{m\in\omega}\rangle \models \varphi\text{”}.$$

$$M \models \text{“}\forall X \subseteq \kappa^{m(\mathbb{X})} \exists Y \subseteq \kappa^{m(\mathbb{Y})} (\langle \kappa, \in, (A_m)_{m\in\omega}\rangle \models \varphi)\text{”}.$$

Let $\pi : M \to N_\alpha$ denote the transitive collapsing map.

$$N_\alpha \models \text{“}\forall X \subseteq \alpha^{m(\mathbb{X})} \exists Y \subseteq \alpha^{m(\mathbb{Y})} (\langle \alpha, \in, (A_m \cap (\alpha^{m(\mathbb{A}_m)}))_{m\in\omega}\rangle \models \varphi)\text{”}.$$

Sakai's forcing

Definition

Let $\mathbb{S}$ be the poset of all pairs $(k, \mathcal{B})$ with the following properties:

1. *k is a function such that dom$(k) < \kappa$;*
2. *for each $\alpha \in$ dom(k), $k(\alpha)$ is a transitive model of ZF^- of size $\leq \max\{\aleph_0, |\alpha|\}$, with $k \restriction \alpha \in k(\alpha)$;*
3. *$\mathcal{B}$ is a subset of $\mathcal{P}(\kappa)$ of size $\leq \mathrm{dom}(k)$;*

$(k', \mathcal{B}') \leqslant (k, \mathcal{B})$ in $\mathbb{S}$ if the following holds:

(i) *$k' \supseteq k$, and $\mathcal{B}' \supseteq \mathcal{B}$;*

(ii) *for any $B \in \mathcal{B}$ and any $\alpha \in$ dom$(k')\setminus$dom(k), $B \cap \alpha \in k'(\alpha)$.*

Fact (Sakai)

For every stationary $S \subseteq \kappa$, $V^{\mathbb{S}} \models \diamondsuit_S^{++}$.

Conclusion

Corollary

For all stationary subsets X and S of κ, there exists a $<\kappa$-closed κ^+-cc forcing extension, in which X $\mathfrak{f}$-reflects to S.

Thank you