Filter Reflection

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Outline

Stationary reflection

Let α be an ordinal of uncountable cofinality. A set $C \subseteq \alpha$ is a club if it is closed and unbounded. A set $S \subseteq \alpha$ is stationary if for all club $C \subset \alpha$, $C \cap S \neq \emptyset$.

Definition

Let κ be a regular uncountable cardinal $\alpha \in \kappa$ be an ordinal of uncountable cofinality, and a stationary $S \subseteq \kappa$, we say that S reflects at α if $S \cap \alpha$ is stationary in α

If κ is a weakly compact cardinal, every stationary subset of κ reflects at a regular cardinal $\alpha < \kappa$.

Generalised descriptive set theory

Suppose κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

The generalised Baire space is the space κ^κ endowed with the bounded topology, for every $\eta \in \kappa^{<\kappa}$ the following set

$$N_\eta = \{\xi \in \kappa^\kappa \mid \eta \subseteq \xi\}$$

is a basic open set.

Equivalence modulo nonstationary

Definition

For every stationary set $S \subseteq \kappa$ and $\theta \in [2, \kappa]$, the equivalence relation $=^\theta_S$ over the subspace θ^κ is defined via

$$\eta =^\theta_S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\} \text{ is non-stationary.}$$

Definition

The quasi-order $\leq^S$ over κ^κ is defined via

$$\eta \leq^S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \text{ is non-stationary.}$$

The quasi-order $\subseteq^S$ over 2^κ is nothing but $\leq^S \cap (2^\kappa \times 2^\kappa)$.

Model Theory and $=^\theta_S$

In model theory, Shelah's main gap theorem can be understood as: *Classifiable theories are less complex than non-classifiable theories.* In generalized descriptive set theory, the complexity of a theory can be study by studying the complexity of the isomorphism relation of the theory. Let λ be a regular cardinal and denote by S^κ_λ the set $\{\alpha < \kappa \mid cf(\alpha) = \lambda\}$. Let us denote by $=^\theta_\lambda$ the relation $=^\theta_S$ when $S = S^\kappa_\lambda$.

Fact (Hyttinen-M)

The isomorphism relation of any classifiable theory is less complex than $=^\kappa_\lambda$ for all λ.

Under some cardinal arithmetic assumptions the following can be proved:

Fact (Friedman-Hyttinen-Kulikov)

Suppose T is a non-classifiable theory. There is a regular cardinal $\lambda < \kappa$ such that $=^2_\lambda$ is as most as complex as the isomorphism relation of T.

Reductions

For $i < 2$, let X_i be some space from the collection $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$. Let R_0 and R_1 be binary relations over X_0 and X_1, respectively.

Definition

A function $f : X_0 \to X_1$ is said to be a reduction of R_0 to R_1 iff, for all $\eta, \xi \in X_0$,

$$\eta \; R_0 \; \xi \text{ iff } f(\eta) \; R_1 \; f(\xi).$$

The existence of a function f satisfying this is denoted by $R_0 \hookrightarrow R_1$.

Lipschitz reductions

For $i < 2$, let X_i be some space from the collection $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$. Let R_0 and R_1 be binary relations over X_0 and X_1, respectively.

For all $\eta, \xi \in \kappa^\kappa$, denote

$$\Delta(\eta, \xi) := \min(\{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cup \{\kappa\}).$$

A reduction f of R_0 to R_1 is said to be 1-Lipschitz iff for all $\eta, \xi \in X_0$,

$$\Delta(\eta, \xi) \leq \Delta(f(\eta), f(\xi)).$$

The existence of a 1-Lipschitz reduction f is denoted by $R_0 \hookrightarrow_1 R_1$. We likewise define $R_0 \hookrightarrow_c R_1$, $R_0 \hookrightarrow_B R_1$ and $R_0 \hookrightarrow_{BM} R_1$ once we replace 1-Lipschitz by a continuous, Borel, or Baire measurable map, respectively.

Comparing $=^\kappa_S$ and $=^2_S$

Fact (Asperó-Hyttinen-Kulikov-M)

If every stationary subset of X reflects at stationary many $\alpha \in Y$, then $=^\kappa_X \hookrightarrow_c =^\kappa_Y$.

Fact (Friedman-Hyttinen-Kulikov)

Suppose $V = L$, and $X \subseteq \kappa$ and $Y \subseteq reg(\kappa)$ are stationary. If every stationary subset of X reflects at stationary many $\alpha \in Y$, then $=^2_X \hookrightarrow_c =^2_Y$.

Limitations

Let λ be a regular cardinal and denote by S^κ_λ the set $\{\alpha < \kappa \mid cf(\alpha) = \lambda\}$.

- For all regular cardinals $\gamma \leqslant \lambda < \kappa$, any $X \subseteq S^\kappa_\lambda$, X does not reflect at any $\alpha \in S^\kappa_\gamma$.
- If $\kappa = \lambda^+$ and $\square_\lambda$ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that Y does not reflect at any $\alpha < \kappa$. This happens in L.
- Usual stationary reflection requires large cardinals.

Outline

The case of L

Recall: $=^\theta_\lambda$ is the relation $=^\theta_S$ when $S = S^\kappa_\lambda$.

Fact (Hyttinen-Kulikov-M)

Suppose $V = L$. Let λ be a regular cardinal below κ. Then for all stationary $X \subseteq \kappa$, $=^\kappa_X \hookrightarrow_c =^2_\lambda$.

Question

How is this possible if there are sets in L that do not reflect at any $\alpha < \kappa$?

Capturing clubs

Suppose S is stationary subset of κ, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α.

Definition

We say that $\vec{\mathcal{F}}$ captures clubs iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is non-stationary;

For any ordinal $\alpha < \kappa$ of uncountable cofinality, denote by $CUB(\alpha)$ the club filter of subsets of α. The sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S^\kappa_{\omega_1} \rangle$ define by $\mathcal{F}_\alpha = CUB(\alpha)$, capture clubs.

Filter reflection

Suppose X and S are stationary subsets of κ, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α.

Definition

We say that X $\vec{\mathcal{F}}$-reflects to S iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary

Definition

We say that X $\mathfrak{f}$-reflects to S iff there exists a sequence of filters $\vec{\mathcal{F}}$ over a stationary subset S' of S such that X $\vec{\mathcal{F}}$-reflects to S'.

Some comments

- Suppose $X, S \subseteq \kappa$ are stationary sets such that every ordinal $\alpha \in S$ has uncountable cofinality and every stationary $Y \subseteq X$ reflects at stationary many $\beta \in S$. Define the sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ by $\mathcal{F}_\alpha = CUB(\alpha)$. Clearly X $\vec{\mathcal{F}}$-reflects to S.

- We call fake reflection the case when X $\mathfrak{f}$-reflects to S and for all $\alpha \in S$, $\mathcal{F}_\alpha \not\supseteq CUB(\alpha)$.

- Suppose $S \subseteq \kappa$ is stationary and $\{S_\beta \mid \beta < \kappa\}$ a partition of S. Define the sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ by: For all $\alpha \in S_\beta$ let $\mathcal{F}_\alpha$ be the filter generated by $\{\beta\}$ if $\beta < \alpha$, and $\{\alpha\}$ otherwise. Clearly for all $Y \subseteq X$, $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary.

Strong forms of filter reflection

Definition

We say that X *strongly* $\vec{\mathcal{F}}$*-reflects to* S *iff* $\vec{\mathcal{F}}$ *captures clubs and, for every stationary* $Y \subseteq X$*, the set* $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha\}$ *is stationary.*

Definition

We say that X $\vec{\mathcal{F}}$*-reflects with* $\diamondsuit$ *to* S *iff* $\vec{\mathcal{F}}$ *captures clubs and there exists a sequence* $\langle Y_\alpha \mid \alpha \in S \rangle$ *such that, for every stationary* $Y \subseteq X$*, the set* $\{\alpha \in S \mid Y_\alpha = Y \cap \alpha \ \& \ Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ *is stationary.*

Outline

Properties

Fact (Monotonicity)

For stationary sets $Y \subseteq X \subseteq \kappa$ and $S \subseteq T \subseteq \kappa$:

1. *If X $\mathfrak{f}$-reflects to S, then Y $\mathfrak{f}$-reflects to T;*
2. *If X strongly $\mathfrak{f}$-reflects to S, then Y strongly $\mathfrak{f}$-reflects to T;*
3. *If X $\mathfrak{f}$-reflects with $\diamondsuit$ to S, then Y $\mathfrak{f}$-reflects with $\diamondsuit$ to T.*

Proposition

Suppose X strongly $\mathfrak{f}$-reflects to S. If $\diamondsuit_X$ holds, then so does $\diamondsuit_S$.

Fake reflection and reductions

Lemma

If X $\mathfrak{f}$-reflects to S, then $=^\kappa_X \hookrightarrow_1 =^\kappa_S$.

Proof.

Suppose that $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S' \rangle$ witnesses that X $\mathfrak{f}$-reflects to S. For every $\alpha \in S'$, define an equivalence relation $\sim_\alpha$ over κ^α by letting $\eta \sim_\alpha \xi$ iff there is $W \in \mathcal{F}_\alpha$ such that $W \cap X \subseteq \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\}$. As there are at most $|\kappa^\alpha|$ many equivalence classes and as $\kappa^{<\kappa} = \kappa$, we may attach to each equivalence class $[\eta]_{\sim_\alpha}$ a unique ordinal (a *code*) in κ, which we shall denote by $\ulcorner [\eta]_{\sim_\alpha} \urcorner$. Next, define a map $f : \kappa^\kappa \to \kappa^\kappa$ by letting for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$:

$$f(\eta)(\alpha) := \begin{cases} \ulcorner [\eta \restriction \alpha]_{\sim_\alpha} \urcorner, & \text{if } \alpha \in S'; \\ 0, & \text{otherwise.} \end{cases}$$

■

Fake reflection and reductions

Lemma

If X strongly $\mathfrak{f}$-reflects to S, then for all $\theta \in [2, \kappa]$, $=^{\theta}_{X} \hookrightarrow_1 =^{\theta}_{S}$.

Lemma

If X $\mathfrak{f}$-reflects with $\diamondsuit$ to S, then $=^{\kappa}_{X} \hookrightarrow_1 =^{2}_{S}$.

What happens in L

Suppose $V = L$. For $\kappa = \lambda^+$, it is known that for all stationary sets $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ that does not reflect at any $\alpha < \kappa$.

Question

What about fake reflection?
Suppose $V = L$. Does X $\mathfrak{f}$-reflects to κ, for all stationary $X \subseteq \kappa$?

A diamond reflection principle

For sets N and x, we say that N *sees* x iff N is a transitive model of ZF^- and $x \cup \{x\} \subseteq N$

Definition

For a stationary $S \subseteq \kappa$ and a positive integer n, $DI^*_S(\Pi^1_n)$ asserts the existence of a sequence $\vec{N} = \langle N_\alpha \mid \alpha \in S \rangle$ satisfying the following:

1. *for every $\alpha \in S$, N_α is a set of cardinality $< \kappa$ that sees α;*
2. *for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$;*
3. *for every Π^1_n-sentence ϕ valid in a structure $\langle \kappa, \in, (A_m)_{m\in\omega} \rangle$, there are stationarily many $\alpha \in S$ such that $|N_\alpha| = |\alpha|$ and*

$$N_\alpha \models \text{“}\phi \text{ is valid in } \langle \alpha, \in, (A_m \restriction \alpha)_{m\in\omega} \rangle\text{”}.$$

$DI^*_S(\Pi^1_1)$ and fake reflection

Lemma

Suppose $S \subseteq \kappa$ is stationary for which $\mathrm{DI}^*_S(\Pi^1_1)$ holds. Then for all stationary $X \subseteq \kappa$, X $\mathfrak{f}$-reflects to S.

Proof.

Idea: Let Φ be a Π^1_1-sentence such that for all α, $\langle \alpha, \in \rangle \models \Phi$ if and only if α is regular. Let $S' \subseteq S$ be the set of ordinals such that $N_\alpha \models$ "Φ *is valid in* $\langle \alpha, \in \rangle$". For all $\alpha \in S'$, define $\mathcal{F}_\alpha$ as the set of $D \in N_\alpha$ such that $N_\alpha \models$ "*D is a club*". ■

Fake reflection in L

Theorem

Suppose $V = L$. For any stationary set $S \subseteq \kappa$, $DI^*_S(\Pi^1_2)$ holds.

Corollary

Suppose $V = L$. Then for every stationary set $S \subseteq \kappa$, κ $\mathfrak{f}$-reflects to S.

Remark

By monotonicity, suppose $V = L$, then for all stationary sets $X, S \subseteq \kappa$, X $\mathfrak{f}$-reflects to S.

In particular S $\mathfrak{f}$-reflects to S and $S^\kappa_{\omega_1}$ $\mathfrak{f}$-reflects to S^κ_ω.

Over the limits

- Usual stationary reflection is a special case of filter reflection.
- For all regular cardinals $\gamma \leqslant \lambda < \kappa$, any $X \subseteq S^\kappa_\lambda$, X does not reflect at any $\alpha \in S^\kappa_\gamma$. S^κ_λ $\mathfrak{f}$-reflects to S^κ_γ is consistently true.
- If $\kappa = \lambda^+$ and $\square_\lambda$ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that Y does not reflect at any $\alpha < \kappa$. Fake reflection is consistent with $\square_\lambda$.
- Fake reflection does not require large cardinals. This is the case of L.

Outline

The Failure

Question

Is the failure of filter reflection consistently true?

- Weakly compact: clearly the failure cannot be forced.
- Usual stationary reflection: force $\square_\lambda$.
- Fake reflection: forcing $\square_\lambda$ is not enough.

Question

What do we need to kill fake reflection?

$I[\kappa - X]$

Definition

Let $X \subseteq \kappa$. We define a collection $I[\kappa - X]$, as follows.
A set Y is in $I[\kappa - X]$ iff $Y \subseteq \kappa$ and there exists a sequence $\langle a_\beta \mid \beta < \kappa \rangle$ of elements of $[\kappa]^{<\kappa}$ along with a club $C \subseteq \kappa$ such that, for every $\delta \in Y \cap C$, there is a cofinal subset $A \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ such that

1. $\{A \cap \gamma \mid \gamma < \delta\} \subseteq \{a_\beta \mid \beta < \delta\}$, *and*
2. $\mathrm{acc}^+(A) \cap X = \emptyset$.

Shelah's *approachability ideal* $I[\kappa]$ is equal to $I[\kappa - \emptyset] \restriction \mathrm{Sing}$

$\mathrm{Add}(\kappa, 1)$

Theorem

Suppose X, S are disjoint stationary subsets of κ, with $S \in I[\kappa - X]$. For every $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$, $V^{\mathrm{Add}(\kappa,1)} \models X$ does not $\vec{\mathcal{F}}$-reflect to S.

Proof: Towards a contradiction, suppose that $\vec{\mathcal{F}}$ is a counterexample.

Let R denote the set of all pairs $(p, q) \in 2^{<\kappa} \times 2^{<\kappa}$ such that:

- $\mathrm{dom}(p) = \mathrm{dom}(q)$ is in $\mathrm{nacc}(\kappa)$;
- $\{\alpha \in \mathrm{dom}(p) \mid p(\alpha) = q(\alpha) = 1\}$ is disjoint from X;
- $\{\alpha \in \mathrm{dom}(q) \mid q(\alpha) = 1\}$ is a closed set of ordinals.

We let $\mathbb{R} := (R, \leq)$ where $(p', q') \leq (p, q)$ iff $p' \supseteq p$ and $q' \supseteq q$.

continuation of the proof

$\mathbb{R}$ is $<\kappa$-closed of size κ, $\mathbb{R}$ is forcing equivalent to $\mathrm{Add}(\kappa, 1)$.

Let $P := \{p \mid \exists q\ (p, q) \in R\}$. It is easy to see that $\mathbb{P} := (P, \supseteq)$ is $<\kappa$-closed, so that $\mathbb{P}$ is forcing equivalent to $\mathrm{Add}(\kappa, 1)$.

Let G be $\mathbb{R}$-generic over V. Let G_0 denote the projection of G to the first coordinate, so that G_0 is $\mathbb{P}$-generic over V.

In $V[G_0]$, let $Q := \{q \in 2^{<\kappa} \mid \exists p \in G_0\ (p, q) \in R\}$. Clearly, $\mathbb{Q} := (Q, \supseteq)$ is isomorphic to the quotient forcing $\mathbb{R}/G_0$.

It follows that, in $V[G]$, we may read a $\mathbb{Q}$-generic set G_1 over $V[G_0]$ such that, in particular, $V[G] = V[G_0][G_1]$.

Denote $\eta := \bigcup G_0$ and let $Y := \{\alpha \in X \mid \eta(\alpha) = 1\}$.

continuation of the proof

Recall: $\eta := \bigcup G_0$ and $Y := \{\alpha \in X \mid \eta(\alpha) = 1\}$
We will prove the following claim:

1. In $V[G_0]$, Y is stationary.
   As $\mathrm{Add}(\kappa, 1)$ is almost homogeneous and $\mathbb{P}$ is equivalent to $\mathrm{Add}(\kappa, 1)$, in $V[G_0]$, X $\vec{\mathcal{F}}$-reflects to S. Therefore, $T := \{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary.
2. In $V[G_0][G_1]$, T is stationary.
3. In $V[G_0][G_1]$, Y is nonstationary.

The last two claims contradict that $\vec{\mathcal{F}}$ capture clubs.

In $V[G_0]$, Y is stationary

Let $\dot{Y}$ be the $\mathbb{P}$-name for Y, that is,

$$\dot{Y} := \{(\check{\alpha}, p) \mid p \in P, \alpha \in X \cap \mathrm{dom}(p), p(\alpha) = 1\}.$$

Let p be an arbitrary condition that $\mathbb{P}$-forces that some $\dot{D}$ is a $\mathbb{P}$-name for a club in κ. Recursively define a sequence $\langle (p_i, \alpha_i) \mid i < \kappa \rangle$ as follows:

- Let (p_0, α_0) be such that $p_0 \supseteq p$ and $p_0 \Vdash_{\mathbb{P}} \check{\alpha}_0 \in \dot{D}$.
- Suppose that $i < \kappa$ for which $\langle (p_j, \alpha_j) \mid j \leqslant i \rangle$ has already been defined. Set $\varepsilon_i := \max\{\alpha_i, \mathrm{dom}(p_i)\} + 1$. Then pick $p_{i+1} \supseteq p_i$ and $\alpha_{i+1} < \kappa$ such that $\varepsilon_i \in \mathrm{dom}(p_{i+1})$ and $p_{i+1} \Vdash_{\mathbb{P}} \check{\alpha}_{i+1} \in \dot{D} \setminus \check{\varepsilon}_i$.

In $V[G_0]$, Y is stationary

- Suppose that $i \in \mathrm{acc}(\kappa)$ and that $\langle (p_j, \alpha_j) \mid j < i \rangle$ has already been defined. Evidently,

$$\sup_{j<i} \varepsilon_j = \sup_{j<i}(\mathrm{dom}(p_j)) = \sup_{j<i} \alpha_j,$$

denote the above common value by α_i . Finally, set $p_i := (\bigcup_{j<i} p_j)^\frown 1$, p_i is a legitimate condition satisfying $\mathrm{dom}(p_i) = \alpha_i + 1$ and $p_i(\alpha_i) = 1$.

Evidently, $E := \{\alpha_i \mid i < \kappa\}$ is a club, we may pick $\beta \in X$ such that $\alpha_\beta = \beta$.

Then $p_\beta \Vdash_{\mathbb{P}} \check{\beta} \in \dot{D} \cap \check{X}$, so that, from $p_\beta(\beta) = 1$, we infer that $p_\beta \Vdash_{\mathbb{P}} \dot{D} \cap \dot{Y} \neq \emptyset$.

In $V[G_0][G_1]$, T is stationary

Fix $\vec{a}$, C in V that witness together that S is in $I[\kappa - X]$. As $\mathbb{P}$ is cofinality-preserving, in $V[G_0]$, the above two still witness together that S is in $I[\kappa - X]$.

Work in $V[G_0]$. As T is a subset of S, $\vec{a}$, C also witness together that T is in $I[\kappa - X]$.

Let q be an arbitrary condition that $\mathbb{Q}$-forces that some $\dot{D}$ is a $\mathbb{Q}$-name for a club in κ.

Fix a large enough regular Θ and some well-ordering $<_\Theta$ of H_Θ; an elementary submodel $N \prec (H_\Theta, <_\Theta)$ such that $\vec{a}, C, \mathbb{Q}, q, \dot{D} \in N$ and $\delta := N \cap \kappa$ is in T.

In $V[G_0][G_1]$, T is stationary

Pick a cofinal subset $A \subseteq \delta$ with $\mathrm{otp}(A) = \mathrm{cf}(\delta)$ and $\mathrm{acc}^+(A) \cap X = \emptyset$ such that:

$$\{A \cap \gamma \mid \gamma < \delta\} \subseteq \{a_\beta \mid \beta < \delta\}.$$

Let $\langle \delta_i \mid i < \mathrm{cf}(\delta) \rangle$ be the increasing enumeration of A.

For every initial segment a of A, we recursively define the following sequence $\langle (q_i, \alpha_i) \mid i \leq \sigma(a) \rangle$, where $\sigma(a)$ will the length of the recursion.

- Let q_0 be the $<_\Theta$-least condition in $\mathbb{Q}$ extending q for which there is $\alpha < \kappa$ such that $q_0 \Vdash_{\mathbb{Q}} \check{\alpha} \in \dot{D}$. Now, let α_0 be the $<_\Theta$-least ordinal α such that $q_0 \Vdash_{\mathbb{Q}} \check{\alpha} \in \dot{D}$.

In $V[G_0][G_1]$, T is stationary

- Suppose that $\langle (q_j, \alpha_j) \mid j \leqslant i \rangle$ has already been defined.

  If $a \setminus \max\{\alpha_i, \mathrm{dom}(q_i), \delta_i\}$ is empty, then we terminate the recursion, and set $\sigma(a) := i$.

  Otherwise, let ε_i be the $<_\Theta$-least element of $a \setminus \max\{\alpha_i, \mathrm{dom}(q_i), \delta_i\}$, and then let q_{i+1} be the $<_\Theta$-least condition in $\mathbb{Q}$ extending q_i satisfying $\varepsilon_i \in \mathrm{dom}(q_{i+1})$ and satisfying that there is $\alpha < \kappa$ such that $q_{i+1} \Vdash_{\mathbb{Q}} \check{\alpha} \in \dot{D} \setminus \varepsilon_i$.

  Now, let α_{i+1} be the $<_\Theta$-least ordinal α such that $q_{i+1} \Vdash_{\mathbb{Q}} \check{\alpha} \in \dot{D} \setminus \varepsilon_i$.

In $V[G_0][G_1]$, T is stationary

- Suppose that i is a limit ordinal and that $\langle (q_j, \alpha_j) \mid j < i\rangle$ has already been defined.

  Evidently,

  $$\sup_{j<i} \varepsilon_j = \sup_{j<i}(\mathrm{dom}(q_j)) = \sup_{j<i} \alpha_j,$$

  so we let α_i denote the above common value.

  As $\{\varepsilon_j \mid j < i\} \subseteq a \subseteq A$ and as $\mathrm{acc}^+(A) \cap X = \emptyset$, we infer that $\alpha_i \notin X$.

  So, $q_i := (\bigcup_{j<i} q_j)^\frown 1$ is a legitimate condition satisfying $\mathrm{dom}(q_i) = \alpha_i + 1$ and $q_i(\alpha_i) = 1$.

In $V[G_0][G_1]$, T is stationary

Recall $\delta := N \cap \kappa$ and $\delta \in T$.

For every $\gamma < \mathrm{cf}(\delta)$, $\langle (q_i, \alpha_i) \mid i \leq \sigma(A \cap \gamma) \rangle$ is in N.

Therefore $\sigma(A) = \mathrm{cf}(\delta)$ and $\alpha_{\mathrm{cf}(\delta)} = \delta$.

Finally $q_{\mathrm{cf}(\delta)} \Vdash_{\mathbb{Q}} \check{\delta} \in \dot{D}$.

In $V[G_0][G_1]$, Y is nonstationary.

Look at the set

$$C := \{\alpha < \kappa \mid \exists q \in G_1(q(\alpha) = 1)\}$$

by the definition of $\mathbb{R}$ and $\mathbb{Q}$, C is a closed subset of κ which is disjoint from Y.

By a density argument we prove that C is unbounded. For every condition q in $\mathbb{Q}$, find $\delta \in S$ above $\mathrm{dom}(q)$. Define $q^\bullet : \delta + 1 \to 2$ via:

$$q^\bullet(\alpha) := \begin{cases} q(\alpha), & \text{if } \alpha \in \mathrm{dom}(q); \\ 1, & \text{if } \alpha = \delta; \\ 0, & \text{otherwise.} \end{cases}$$

$\{\alpha < \kappa \mid q^\bullet(\alpha) = 1\}$ is a proper end-extension of $\{\alpha < \kappa \mid q(\alpha) = 1\}$.

Killing fake reflection

Corollary

Suppose X, S are disjoint stationary subsets of κ, with $S \in I[\kappa - X]$. After forcing with $\mathrm{Add}(\kappa, \kappa^+)$, X does not $\mathfrak{f}$-reflect to S.

By doing a preliminary forcing to enlarge $I[\kappa - X]$ for all X, we obtain:

Corollary (Dense non-reflection)

There exists a cofinality-preserving forcing extension in which for all two disjoint stationary subsets X, S of κ, X does not $\mathfrak{f}$-reflect to S.

Killing fake reflection

Lemma

Suppose that κ is strongly inaccessible or $\kappa = \lambda^+$ with $\lambda^{<\lambda} = \lambda$. For every stationary $X, Y \subseteq \kappa$ such that $\mathrm{Tr}(X) \cap Y$ is non-stationary, $Y \in I[\kappa - X]$.

Corollary

If κ is strongly inaccessible (e.g., κ Laver-indestructible supercompact), then in the forcing extension by $\mathrm{Add}(\kappa, \kappa^+)$, for all two disjoint stationary subsets X, S of κ, the following are equivalent:

1. *X $\mathfrak{f}$-reflects to S;*
2. *every stationary subset of X reflects in S.*

Theorem

Martin's Maximum The consistency of Martin's Maximum implies the consistency of

$$=^{\kappa}_{\omega} \hookrightarrow_1 =^2_{\omega_1}$$

by an elaboration on the proof to kill fake reflection we can furthermore kill the reduction:

$$=^2_{\omega_1} \not\hookrightarrow_{BM} =^{\kappa}_{\omega} .$$

Outline

The next step

Question

Can we force filter reflection?

Easy answer: Yes. Just force usual stationary reflection (collapse a weakly compact cardinal).

Question

Can we force fake reflection without using large cardinals?

Sakai's $\diamondsuit^{++}$

Definition

For a stationary $S \subseteq \kappa$, $\diamondsuit_S^{++}$ asserts the existence of a sequence $\langle K_\alpha \mid \alpha \in S \rangle$ satisfying the following:

1. *for every infinite $\alpha \in S$, K_α is a set of size $|\alpha|$;*
2. *for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $C \cap \alpha, X \cap \alpha \in K_\alpha$;*
3. *the following set is stationary in $[H_{\kappa^+}]^{<\kappa}$:*

$$\{M \in [H_{\kappa^+}]^{<\kappa} \mid M \cap \kappa \in S \ \&\ \mathrm{clps}(M, \in) = (K_{M\cap\kappa}, \in)\}.$$

$\diamondsuit^{++}$ and $DI^*_S(\Pi^1_n)$

Recall: For sets N and x, we say that N *sees* x iff N is a transitive model of ZF^- and $x \cup \{x\} \subseteq N$.

Lemma

For every stationary $S \subseteq \kappa$, $\diamondsuit^{++}_S$ implies $DI^*_S(\Pi^1_2)$.

Proof (sketch): Suppose $\langle K_\alpha \mid \alpha \in S\rangle$ is a $\diamondsuit^{++}_S$-sequence. Define a sequence $\vec{N} = \langle N_\alpha \mid \alpha \in S\rangle$ by letting $N_\alpha = K_\alpha$ if K_α sees α, otherwise N_α is any model of ZF^- that sees α and contains K_α. Let $\phi = \forall X \exists Y \varphi$ be a Π^1_2-sentence and $(A_m)_{m\in\omega}$ be such that $\langle \kappa, \in, (A_m)_{m\in\omega}\rangle \models \phi$. Given an arbitrary club $C \subseteq \kappa$, we consider the following set

$$\mathcal{C} := \{M \prec H_{\kappa^+} \mid M \cap \kappa \in C \;\&\; (A_m)_{m\in\omega} \in M\}.$$

$\mathcal{C}$ is a club in $[H_{\kappa^+}]^{<\kappa}$.

$\diamondsuit^{++}$ and $DI^*_S(\Pi^1_n)$

Lemma

For every stationary $S \subseteq \kappa$, $\diamondsuit^{++}_S$ implies $DI^*_S(\Pi^1_2)$.

Proof continuation (sketch): By $\diamondsuit^{++}_S$ the set

$$\mathcal{C} \cap \{M \in [H_{\kappa^+}]^{<\kappa} \mid M \cap \kappa \in S \ \& \ \text{clps}(M, \in) = (K_{M\cap\kappa}, \in)\}$$

is stationary, pick M in this set. Since $\langle \kappa, \in, (A_m)_{m\in\omega}\rangle \models \phi$, by definition

$$H_{\kappa^+} \models \text{“}\forall X \subseteq \kappa^{m(\mathbb{X})} \ \exists Y \subseteq \kappa^{m(\mathbb{Y})} \ \langle \kappa, \in, (A_m)_{m\in\omega}\rangle \models \varphi\text{”}.$$

$$M \models \text{“}\forall X \subseteq \kappa^{m(\mathbb{X})} \exists Y \subseteq \kappa^{m(\mathbb{Y})} (\langle \kappa, \in, (A_m)_{m\in\omega}\rangle \models \varphi)\text{”}.$$

Let $\pi : M \to N_\alpha$ denote the transitive collapsing map.

$$N_\alpha \models \text{“}\forall X \subseteq \alpha^{m(\mathbb{X})} \exists Y \subseteq \alpha^{m(\mathbb{Y})} (\langle \alpha, \in, (A_m \cap (\alpha^{m(\mathbb{A}_m)}))_{m\in\omega}\rangle \models \varphi)\text{”}.$$

Sakai's forcing

Definition

Let $\mathbb{S}$ be the poset of all pairs $(k, \mathcal{B})$ with the following properties:

1. *k is a function such that dom$(k) < \kappa$;*
2. *for each $\alpha \in$ dom(k), $k(\alpha)$ is a transitive model of ZF^- of size $\leq \max\{\aleph_0, |\alpha|\}$, with $k \restriction \alpha \in k(\alpha)$;*
3. *$\mathcal{B}$ is a subset of $\mathcal{P}(\kappa)$ of size $\leq \mathrm{dom}(k)$;*

$(k', \mathcal{B}') \leqslant (k, \mathcal{B})$ in $\mathbb{S}$ if the following holds:

(i) *$k' \supseteq k$, and $\mathcal{B}' \supseteq \mathcal{B}$;*

(ii) *for any $B \in \mathcal{B}$ and any $\alpha \in$ dom$(k')\setminus$dom(k), $B \cap \alpha \in k'(\alpha)$.*

Fact (Sakai)

For every stationary $S \subseteq \kappa$, $V^{\mathbb{S}} \models \diamondsuit_S^{++}$.

Conclusion

Corollary

For all stationary subsets X and S of κ, there exists a $<\kappa$-closed κ^+-cc forcing extension, in which X $\mathfrak{f}$-reflects to S.

Thank you