Colouring orders and ordering trees

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The spectrum problem

Let $I(T, \alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T, \alpha)$?

- **Löwenheim-Skolem Theorem:** $\exists\alpha \geq \omega\ I(T, \alpha) \neq 0 \Rightarrow \forall\beta \geq \omega\ I(T, \beta) \neq 0$.
- **Morley's categoricity:** $\exists\alpha > \omega\ I(T, \alpha) = 1 \Rightarrow \forall\beta > \omega\ I(T, \beta) = 1$
- **Shelah's Main Gap Theorem:** Either, for every uncountable cardinal α, $I(T, \alpha) = 2^\alpha$, or $\forall\alpha > 0\ I(T, \aleph_\alpha) < \beth_{\omega_1}(|\alpha|)$.

Approaches

- Shelah's stability theory.
  Classify the models of T by different divisible lines that clearly differentiate between the theories that can be classified and those that cannot.

- Descriptive set theory:
  It uses Borel-reducibility and the isomorphism relation to define a partial order on the set of all first-order complete countable theories.

The topology

κ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

Coding structures

Fix a language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in \kappa^\kappa$ define the structure $\mathcal{A}_f$ with domain κ and for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

Definition (The isomorphism relation)

Given T a first-order complete countable theory in a countable vocabulary, we say that $f, g \in \kappa^\kappa$ are $\cong_T^\kappa$ equivalent if
$\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
or $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x,y)\in E_1 \Leftrightarrow (f(x),f(y))\in E_2$.
We write $E_1 \hookrightarrow_b^\kappa E_2$.

If the function is continuous, then we say that E_1 is *continuous reducible* to E_2 and we denote it by $E_1 \hookrightarrow_c^\kappa E_2$.

We can define a partial order on the set of all first-order complete countable theories

$$T \leq^\kappa T' \text{ iff } \cong_T^\kappa \hookrightarrow_c^\kappa \cong_{T'}^\kappa$$

Shelah's Main Gap Theorem

Theorem (Shelah)

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Question:

Is there a continuous reducibility counterpart of the Main Gap Theorem in the spaces κ^κ?

non-classifiable theories

A theory T is non-classifiable if it is a countable complete theory that satisfies one of the following:

- T is unstable;
- T is stable unsuperstable;
- T is superstable with DOP;
- T is superstable with OTOP.

Progress

Theorem (Friedman - Hyttinen - Kulikov)

If T is classifiable and T' is unsuperstable, then

$$T' \not\leq^{\kappa} T$$

Theorem (Hyttinen - Moreno)

Suppose T is a classifiable theory, T' is an stable theory with the OCP, and κ an inaccessible cardinal. Then

$$T \leq^{\kappa} T'$$

Progress

Theorem (Moreno)

Suppose T is a classifiable theory, T' is a superstable theory with the S-DOP, and κ an inaccessible cardinal. Then

$$T \leq^\kappa T'$$

Theorem (Mangraviti - Motto Ros)

Let $\kappa = \aleph_\gamma$ be such that $\kappa^{<\kappa} = \kappa$ and $\beth_{\omega_1}(|\gamma|) \leq \kappa$. Let T, T' be countable complete first-order theories, and suppose T is classifiable and shallow, while T' is not. Then

$$\cong^\kappa_T \hookrightarrow^\kappa_b \cong^\kappa_{T'}$$

Progress

Theorem (Hyttinen - Kulikov - Moreno)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, and $\lambda^{<\lambda} = \lambda$. There is a κ-closed κ^+-cc forcing which forces:
If T is classifiable and T' is not, then $T \leq^\kappa T'$ and $T' \not\leq^\kappa T$

Theorem (Fernandes - Moreno - Rinot)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, and $\lambda^{<\lambda} = \lambda$. Let T be a non-classifiable theory. There is a κ-closed κ^+-cc forcing which forces:
If T' is a countable complete first-order theory, then $T' \leq^\kappa T$.

Stable unsuperstable theories

Theorem (Hyttinen - Kulikov - Moreno)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, and $\lambda^{<\lambda} = \lambda$. If T is classifiable and T' is stable unsuperstable, then $T \leq^\kappa T'$.

Ordered trees

Definition

Let K^{ω}_{tr} be the class of models $(A, \prec, (P_n)_{n\leq\omega}, <, h)$, where:

- there is a linear order $(I, <_I)$ such that $A \subseteq I^{\leq\omega}$;
- A is closed under initial segment;
- $\prec$ is the initial segment relation;
- $h(\eta, \xi)$ is the maximal common initial segment of η and ξ;
- let $lg(\eta)$ be the length of η (i.e. the domain of η) and $P_n = \{\eta \in A \mid lg(\eta) = n\}$ for $n \leq \omega$;

Ordered trees

Definition (continuation)

Let K^{ω}_{tr} be the class of models $(A, \prec, (P_n)_{n\leq\omega}, <, h)$, where:

- for every $\eta \in A$ with $lg(\eta) < \omega$, define $Suc_A(\eta)$ as $\{\xi \in A \mid \eta \prec \xi \wedge lg(\xi) = lg(\eta) + 1\}$. If $\xi < \zeta$, then there is $\eta \in A$ such that $\xi, \zeta \in Suc_A(\eta)$;
- for every $\eta \in A \backslash P_\omega$, $<\restriction Suc_A(\eta)$ is the induced linear order from I, i.e.
$$\eta^\frown\langle x\rangle < \eta^\frown\langle y\rangle \Leftrightarrow x <_I y;$$
- If η and ξ have no immediate predecessor and $\{\zeta \in A \mid \zeta \prec \eta\} = \{\zeta \in A \mid \zeta \prec \xi\}$, then $\eta = \xi$.

Generalized Ehrenfeucht-Mostowski models

Definition

Suppose T is a countable complete theory in a countable vocabulary $\mathcal{L}$, $\mathcal{L}^1$ a Skolemization of $\mathcal{L}$, and T^1 the Skolemization of T by $\mathcal{L}^1$.

We say that a function Φ is proper for K^{ω}_{tr}, if for each $A \in K^{\omega}_{tr}$, there is a model $\mathcal{M}_1$ and tuples a_s, $s \in A$, of elements of $\mathcal{M}_1$ such that the following two hold:

- every element of $\mathcal{M}_1$ is an interpretation of some $\mu(a_s)$, where μ is a $\mathcal{L}^1$-term;
- $tp_{at}(a_s, \emptyset, \mathcal{M}_1) = \Phi(tp_{at}(s, \emptyset, A))$.

Proper function

We denote $\mathcal{M}_1$ by $EM^1(A,\Phi)$. Denote by $\mathrm{EM}(A,\Phi)$ the $\mathcal{L}$-reduction of $EM^1(A,\Phi)$

Theorem (Shelah)

Suppose $\mathcal{L} \subseteq \mathcal{L}^1$ are vocabularies, T is a complete first order theory in $\mathcal{L}$, T^1 is a complete theory in $\mathcal{L}^1$ extending T and with Skolem-functions.
Suppose T is unsuperstable and $\{\phi_n(x,y_n) \mid n<\omega\}$ witnesses this. Then there is a function Φ proper such that for all $A \in K^{\omega}_{tr}$, $EM^1(A,\Phi)$ is a model of T^1, and for $s \in P^A_n$, $t \in P^A_\omega$, $EM^1(A,\Phi) \models \phi_n(a_t,a_s)$ if and only if $A \models s \prec t$.

κ-representation

Definition

Let A be an arbitrary set of size at most κ. A sequence $\mathbb{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ is a κ-representation of A, if $\langle A_\alpha \mid \alpha < \kappa \rangle$ is an increasing continuous sequence of subsets of A, for all $\alpha < \kappa$, $|A_\alpha| < \kappa$, and $\bigcup_{\alpha<\kappa} A_\alpha = A$.

$S(\mathbb{A})$

Definition

For any $A \in K^{\omega}_{tr}$ with size κ and $\mathbb{A}$ a κ-representation of A, we define $S(\mathbb{A})$ as the set of limit ordinals $\delta < \kappa$ for which exists $\eta \in P^{A}_{\omega}$ such that the following hold

- $\{\eta \restriction n \mid n < \omega\} \subseteq A_\delta$
- $\forall \alpha < \delta \ (\{\eta \restriction n \mid n < \omega\} \not\subseteq A_\alpha)$

Equivalence modulo S

Definition

Given $S \subseteq \kappa$, we define the equivalence relation $=^2_S \subseteq 2^\kappa \times 2^\kappa$, as follows

$$\eta =^2_S \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

We will denote by $=^2_\mu$ the relation $=^2_S$ when $S = \{\alpha < \kappa \mid cf(\alpha) = \mu\}$.

The isomorphism

Theorem (Shelah)

Suppose T is a countable complete unsuperstable theory in a countable vocabulary.
If κ is a regular uncountable cardinal, $A_1, A_2 \in K^{\omega}_{tr}$ have size κ, A_1, A_2 are locally (κ, bs, bs)-nice and $(<\kappa, bs)$-stable, $EM(A_1, \Phi)$ is isomorphic to $EM(A_2, \Phi)$, then $S(A_1) =^2_\omega S(A_2)$.

The goal

Theorem (Hyttinen - Kulikov - Moreno)

Assume T is a countable complete classifiable theory over a countable vocabulary. If $\diamondsuit_\omega$ holds, then $\cong^\kappa_T \hookrightarrow^\kappa_c =^2_\omega$.

The Objective $=^2_\omega \hookrightarrow^\kappa_c \cong^\kappa_T$ for any T unsuperstable.

For all $\eta \in 2^\kappa$ contruct an ordered tree A_η such that the following hold:

- construct them in a smooth way, i.e. the obtained reduction is continuous;
- for all $\eta, \xi \in 2^\kappa$, $\eta =^2_\omega \xi$ if and only if $A_\eta \cong A_\xi$;
- for all $\eta, \xi \in 2^\kappa$, $\eta =^2_\omega \xi$ if and only if $S(A_\eta) =^2_\omega S(A_\xi)$;
- A_η is locally (κ, bs, bs)-nice;
- A_η is $(< \kappa, bs)$-stable;

Colored trees

Definition

A coloured tree is a pair (t, c), where t is a κ^+, $(\omega + 2)$-tree and c is a map $c : t_\omega \to 2$, where t_ω i the set of leaves.

Theorem (Hyttinen - Kulikov)

It is possible to construct for any $f \in 2^\kappa$ a colored tree J_f such that: For every $f, g \in 2^\kappa$ the following holds

$$f \ =^2_\omega \ g \Leftrightarrow J_f \cong J_g$$

Coloring orders

Definition

Let I be a linear order of size κ. We say that I is κ-colorable if there is a function $F : I \to \kappa$ such that for all $B \subseteq I$, $|B| < \kappa$, $b \in I \backslash B$, and $p = tp_{bs}(b, B, I)$ such that the following hold: For all $\alpha \in \kappa$, $|\{a \in I \mid a \models p \;\&\; F(a) = \alpha\}| = \kappa$.

Theorem

Suppose I is a κ-colorable linear order. Then for any $f \in 2^\kappa$, there is an ordered coloured tree $A_f(I)$ that satisfies:
For all $f, g \in 2^\kappa$,

$$f \;=^2_\omega\; g \;\Leftrightarrow A_f(I) \cong A_g(I),$$

$$\text{and } S(A_f(I)) = \{\delta < \kappa \mid cf(\delta) = \omega \wedge f(\delta) = 1\}.$$

Recall

The Objective $=^2_\omega \hookrightarrow^\kappa_c \cong^\kappa_T$ for any T unsuperstable.
For all $\eta \in 2^\kappa$ contruct I is a κ-colorable linear order such that the following hold:

- $A_\eta(I)$ is locally (κ, bs, bs)-nice;
- $A_\eta(I)$ is $(< \kappa, bs)$-stable;

Nice linear order

Definition (Lemma by Hyttinen - Tuuri)

Let I be a linear order of size κ and $\langle I_\alpha \mid \alpha < \kappa\rangle$ a κ-representation. Then I is (κ, bs, bs)-nice if the following hold: There is a club $C \subseteq \kappa$, such that for all limit $\delta \in C$, for all $x \in I$ there is $\beta < \delta$ such that one of the following holds:

- $\forall\sigma \in I_\delta[\sigma \geq x \Rightarrow \exists\sigma' \in I_\beta\ (\sigma \geq \sigma' \geq x)]$
- $\forall\sigma \in I_\delta[\sigma \leq x \Rightarrow \exists\sigma' \in I_\beta\ (\sigma \leq \sigma' \leq x)]$

Locally nice ordered tree

Definition

$A \in K^{\omega}_{tr}$ of size at most κ, is locally (κ, bs, bs)-nice if for every $\eta \in A \backslash P^A_\omega$, $(Suc_A(\eta), <)$ is (κ, bs, bs)-nice, $Suc_A(\eta)$ is infinite, and there is $\xi \in P^A_\omega$ such that $\eta \prec \xi$.

Stable ordered tree

Definition

$A \in K^{\omega}_{tr}$ is $(<\kappa, bs)$-stable if for every $B \subseteq A$ of size smaller than κ,

$$\kappa > |\{tp_{bs}(a, B, A) \mid a \in A\}|.$$

Initial order

Definition

Let $\mathbb{Q}$ be the linear order of the rational numbers.
Let $\kappa \times \mathbb{Q}$ be order by the lexicographic order, I^0 be the set of functions $f : \omega \to \kappa \times \mathbb{Q}$ such that $f(n) = (f_1(n), f_2(n))$, for which $\{n \in \omega \mid f_1(n) \neq 0\}$ is finite.
If $f, g \in I^0$, then $f < g$ if and only if $f(n) < g(n)$, where n is the least number such that $f(n) \neq g(n)$.

Initial order

Lemma

There is a κ-representation $\langle I^0_\alpha \mid \alpha < \kappa\rangle$ such that for all limit $\delta < \kappa$ and $\nu \in I^0$ there is $\beta < \delta$ which satisfies the following:

$$\forall \sigma \in I^0_\delta[\sigma > \nu \Rightarrow \exists \sigma' \in I^0_\beta \ (\sigma \geq \sigma' \geq \nu)]$$

In particular

There is a κ-representation $\langle I^0_\alpha \mid \alpha < \kappa\rangle$ such that for all limit $\delta < \kappa$ and $\nu \in I^0$, if $\nu \notin I^0_\delta$ there is $\beta < \delta$ which satisfies the following:

$$\forall \sigma \in I^0_\delta[\sigma > \nu \Rightarrow \exists \sigma' \in I^0_\beta \ (\sigma > \sigma' > \nu)]$$

Proof

For all $\gamma < \kappa$, let us define $\langle I^0_\alpha \mid \alpha < \kappa \rangle$ by

$$I^0_\gamma = \{\nu \in I^0 \mid \nu_1(n) < \gamma \textit{ for all } n < \omega\}$$

it is clear that $\langle I^0_\alpha \mid \alpha < \kappa \rangle$ is a κ-representation.
Suppose $\delta < \kappa$ is a limit and $\nu \in I^0$. If $\nu \in I^0_\delta$, then there is $\beta < \delta$ such that $\nu \in I^0_\beta$ and the result follows.
Let us take care of the case $\nu \notin I^0_\delta$. Let $\beta < \delta$ be the least ordinal such that for all $n < \omega$, $\nu_1(n) < \delta$ implies $\nu_1(n) < \beta$.

Proof

Claim: For all $\sigma \in I^0_\delta$. If $\sigma > \nu$, then there is $\sigma' \in I^0_\beta$ such that $\sigma \neq \sigma'$ and $\sigma > \sigma' > \delta$.

Proof of the claim: Let us suppose $\sigma \in I^0_\delta$ is such that $\sigma \geq \nu$.
By the definition of I^0, there is $n < \omega$ such that $\sigma(n) > \nu(n)$ and n is the minimum number such that $\sigma(n) \neq \nu(n)$.
Since $\sigma \in I^0_\delta$, for all $m \leq n$, $\nu_1(m) \leq \sigma_1(m) < \delta$. Thus for all $m \leq n$, $\nu_1(m) < \beta$.
Let us divide the proof in two cases, $\sigma_1(n) = \nu_1(n)$ and $\sigma_1(n) > \nu_1(n)$.

Proof

Case 1. $\sigma_1(n) = \nu_1(n)$.
By the density of $\mathbb{Q}$ there is r such that $\sigma_2(n) > r > \nu_2(n)$.
Let us define σ' by:

$$\sigma'(m) = \begin{cases} \nu(m) & \text{if } m < n \\ (\nu_1(n), r) & \text{if } m = n \\ 0 & \text{in other case.} \end{cases}$$

Case 2. $\sigma_1(n) > \nu_1(n)$.
Let us define σ' by:

$$\sigma'(m) = \begin{cases} \nu(m) & \text{if } m < n \\ (\nu_1(n), \nu_2(n)+1) & \text{if } m = n \\ 0 & \text{in other case.} \end{cases}$$

Clearly $\sigma > \sigma' > \nu$. Since $\nu_1(m) < \beta$ for all $m \leq n$, $\sigma' \in I^0_\beta$.

The orders

Suppose $i < \kappa$ is such that I^i has been defined.
For all $\nu \in I^i$ let ν^{i+1} be such that

$$\nu^{i+1} \models tp_{bs}(\nu, I^i \backslash \{\nu\}, I^i) \cup \{\nu > x\}.$$

Notice that ν^{i+1} is a copy of ν that is smaller than ν.
Let $I^{i+1} = I^i \cup \{\nu^{i+1} \mid \nu \in I^i\}$.
Suppose $i < \kappa$ is a limit ordinal such that for all $j < i$, I^j has been defined, we define I^i by $I^i = \bigcup_{j<i} I^j$.

The representations

Suppose $i < \kappa$ is such that $\langle I^i_\alpha \mid \alpha < \kappa\rangle$ has been defined.
For all $\alpha < \kappa$,

$$I^{i+1}_\alpha = I^i_\alpha \cup \{\nu^{i+1} \mid \nu \in I^i_\alpha\}.$$

Suppose $i < \kappa$ is a limit ordinal such that for all $j < i$, $\langle I^j_\alpha \mid \alpha < \kappa\rangle$ has been defined, we define $\langle I^i_\alpha \mid \alpha < \kappa\rangle$ by

$$I^i_\alpha = \bigcup_{j<i} I^j_\alpha.$$

The order

Let us define I as

$$I = \bigcup_{j<\kappa} I^j$$

and the κ-representation $\langle I_\alpha \mid \alpha < \kappa \rangle$ as

$$I_\alpha = \bigcup_{\alpha<\kappa} I^\alpha_\alpha.$$

A different perspective

Definition (Generator)

For all $\nu \in I$ let us denote by $o(\nu)$ the least ordinal $\alpha < \kappa$ such that $\nu \in I^\alpha$.

Let us denote the generator of ν by $Gen(\nu)$ and define it by induction as follows:

- $Gen^i(\nu) = \emptyset$, for all $i < o(\nu)$;
- $Gen^i(\nu) = \{\nu\}$, for $i = o(\nu)$;
- for all $i \geq o(\nu)$,

$$Gen^{i+1}(\nu) = Gen^i(\nu) \cup \{\sigma \in I^{i+1} \mid \exists \tau \in Gen^i(\nu)\ [\tau^{i+1} = \sigma]\};$$

- for all $i < \kappa$ limit,

$$Gen^i(\nu) = \bigcup Gen^j(\nu).$$

A different perspective

Finally, let

$$Gen(\nu) = \bigcup_{i<\kappa} Gen^i(\nu).$$

Suppose $\nu \in I$. For all $\sigma \in Gen(\nu)$, $\sigma \neq \nu$, there is $n < \omega$ and a sequence $\{\sigma_i\}_{i \leq n}$ such that the following holds:

- $\sigma_0 = \nu$;
- for all $j < n$,
$$\sigma_{j+1} = (\sigma_j)^{o(\sigma_{j+1})};$$
- $\sigma = \sigma_n = (\sigma_{n-1})^{o(\sigma)}$

Nice property I^i

Lemma

For all $i < \kappa$, $\delta < \kappa$ a limit ordinal, and $\nu \in I^i$, there is $\beta < \delta$ that satisfies the following:

$$\forall \sigma \in I^i_\delta \; [\sigma > \nu \Rightarrow \exists \sigma' \in I^i_\beta \; (\sigma \geq \sigma' \geq \nu)]$$

In particular. If $\nu \notin I^i_\delta$ there is $\beta < \delta$ which satisfies the following:

$$\forall \sigma \in I^i_\delta [\sigma > \nu \Rightarrow \exists \sigma' \in I^0_\beta \; (\sigma > \sigma' > \nu)]$$

Nice property I

Lemma

For all $\delta < \kappa$ a limit ordinal, and $\nu \in I$, there is $\beta < \delta$ that satisfies the following:

$$\forall \sigma \in I_\delta \; [\sigma > \nu \Rightarrow \exists \sigma' \in I_\beta \; (\sigma \geq \sigma' \geq \nu)]$$

$(<\kappa, bs)$-stable I^0

Theorem (Hyttinen - Tuuri)

Let $\mathcal{R}$ be the set of functions $f:\omega\to\kappa$ for which $\{n\in\omega \mid f(n)\neq 0\}$ is finite. If $f,g\in\mathcal{R}$, then $f<g$ if and only if $f(n)<g(n)$, where n is the least number such that $f(n)\neq g(n)$. If $\lambda^\omega=\lambda$, then the linear order $\mathcal{R}$ is $(<\kappa, bs)$-stable.

Lemma

Suppose $\kappa=\lambda^+$ and $\lambda^\omega=\lambda$. I^0 is $(<\kappa, bs)$-stable.

Proof

For all $A \subseteq I^0$ define $Pr(A)$ as the set $\{f_1 \mid f \in A\}$. Let $A \subseteq I^0$ be such that $|A| < \kappa$.
Since $|\mathbb{Q}| = \omega$,
$|\{tp_{bs}(a, A, I^0) \mid a \in I^0\}| \leq |\{tp_{bs}(a, Pr(A), \mathcal{R}) \mid a \in \mathcal{R}\} \times 2^\omega|$.
Since $\lambda^\omega = \lambda$, $|\{tp_{bs}(a, A, I^0) \mid a \in I\}| < \kappa$.

$(<\kappa, bs)$-stable I

Lemma

Suppose $\kappa=\lambda^+$ and $\lambda^\omega=\lambda$. I is $(<\kappa, bs)$-stable.

Theorem

I is a $(<\kappa, bs)$-stable (κ, bs, bs)-nice κ-colorable linear order.

Corollary

Theorem

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^\omega = \lambda$. If T_1 is a countable complete classifiable theory, and T_2 is a countable complete unsuperstable theory, then $T_1 \ \leq^\kappa \ T_2$.

Theorem

There exists a $<\kappa$-closed κ^+-cc forcing extension in which for all countable complete unsuperstable theory T, $\cong^\kappa_T$ is Σ^1_1-complete.

Thank you