The equivalence modulo non-stationary ideals

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Outline

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The spectrum problem

Let $I(T, \alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T, \alpha)$?

- **Löwenheim-Skolem Theorem:** $\exists \alpha \geq \omega \ I(T, \alpha) \neq 0 \Rightarrow \forall \beta \geq \omega \ I(T, \beta) \neq 0$.
- **Morley's categoricity:** $\exists \alpha > \omega \ I(T, \alpha) = 1 \Rightarrow \forall \beta > \omega \ I(T, \beta) = 1$
- **Shelah's Main Gap Theorem:** Either, for every uncountable cardinal α, $I(T, \alpha) = 2^\alpha$, or $\forall \alpha > 0 \ I(T, \aleph_\alpha) < \beth_{\omega_1}(\mid \alpha \mid)$.

Approaches

- Shelah's stability theory.
  Classify the models of T by cardinal invariants and clearly differentiate between the theories that can be classified and those that cannot.

- Descriptive set theory:
  It uses Borel-reducibility and the isomorphism relation to define a partial order on the set of all first-order complete countable theories.

The topology

κ is a cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set 2^κ with the bounded topology. For every $\zeta \in 2^{<\kappa}$, the set

$$[\zeta] = \{\eta \in 2^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

The collection of Borel subsets of 2^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ.

Reductions

A function $f\colon 2^\kappa \to 2^\kappa$ is *Borel*, if for every open set $A \subseteq 2^\kappa$ the inverse image $f^{-1}[A]$ is a Borel subset of 2^κ.

Let E_1 and E_2 be equivalence relations on 2^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f\colon 2^\kappa \to 2^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

We write $E_1 \leqslant_B E_2$.

Coding structures

Fix a language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in 2^\kappa$ define the structure $\mathcal{A}_f$ with domain κ by: for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \ldots, a_n)) = 1$$

Definition (The isomorphism relation)

Given T a first-order complete countable theory in a countable vocabulary, we say that $f, g \in 2^\kappa$ are $\cong^\kappa_T$ equivalent if

- $\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
  or
- $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

The complexity

We can define a partial order on the set of all first-order complete countable theories

$$T \leqslant^{\kappa} T' \textit{ iff } \cong^{\kappa}_{T} \leqslant_{B} \cong^{\kappa}_{T'}$$

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Shelah's Main Gap Theorem

Theorem (Shelah)

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Question:
Is there a Borel reducibility counterpart of the Main Gap Theorem in the space 2^κ?

Countable

$T = Th(\mathbb{Q}, \leqslant)$.
T', the theory of vector space over the field of rational numbers.

By the Borel-reducibility hierarchy:

$$T \leqslant^{\omega} T'$$

$$T' \not\leqslant^{\omega} T$$

By the stability theory T' is simpler than T.

Uncountable

Theorem (Shelah)

If T is classifiable, then $\cong^{\kappa}_{T}$ is Δ^1_1.

Theorem (S. Friedman, Hyttinen, Kulikov)

If T is unstable then $\cong^{\kappa}_{T}$ is not Δ^1_1.

Theorem (S. Friedman, Hyttinen, Kulikov)

If T is unstable and T' is classifiable, then $T \not\leq^{\kappa} T'$.

The Equivalence Modulo Non-stationary Ideals

Definition

For every $X \subset \kappa$ stationary, we define E^2_X as the relation

$$E^2_X = \{(\eta,\xi) \in 2^\kappa \times 2^\kappa \mid (\eta^{-1}[1] \triangle \xi^{-1}[1]) \cap X \textit{ is not stationary}\}$$

where $\triangle$ denotes the symmetric difference.

When $X = \{\alpha < \kappa | cf(\alpha) = \lambda\}$, we will denote E^2_X by E^2_λ.

Looking above the Gap

Theorem (S. Friedman, Hyttinen, Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$.

- *If T is an unstable or superstable with OTOP, then $E^2_\lambda \leqslant_B \cong^\kappa_T$.*
- *If $\lambda \geq 2^\omega$ and T is a superstable with DOP, then $E^2_\lambda \leqslant_B \cong^\kappa_T$.*

Theorem (S. Friedman, Hyttinen, Kulikov)

Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and T is a stable unsuperstable. Then $E^2_\omega \leqslant_B \cong^\kappa_T$

Looking below the Gap

Theorem (S. Friedman, Hyttinen, Kulikov)

If T is a classifiable theory, then for all regular cardinal $\lambda < \kappa$, $E^2_\lambda \not\leqslant_B \cong^\kappa_T$

Theorem (Hyttinen, Kulikov, M.)

Denote by S^κ_λ the set $\{\alpha < \kappa | cf(\alpha) = \lambda\}$.
Suppose T is a classifiable theory and $\lambda < \kappa$ is a regular cardinal. If $\diamondsuit(S^\kappa_\lambda)$ holds, then $\cong^\kappa_T \leqslant_B E^2_\lambda$.

A Generalized Borel-reducibility Counterpart

Theorem (Hyttinen, Kulikov, M.)

Suppose $\kappa=\lambda^+$ and $\lambda^\omega=\lambda$. If T is a classifiable theory and T' is a stable unsuperstable theory, then $\cong^\kappa_T \leqslant_B E^2_\omega \leqslant_B \cong^\kappa_{T'}$ and $E^2_\omega \not\leqslant_B \cong^\kappa_T$.

Let $H(\kappa)$ be the following property: If T is classifiable and T' is not, then $T \leqslant^\kappa T'$ and $T' \not\leqslant^\kappa T$.

Theorem (Hyttinen, Kulikov, M.)

Suppose $\kappa=\lambda^+$, $2^\lambda>2^\omega$ and $\lambda^{<\lambda}=\lambda$. If $V=L$, then $H(\kappa)$ holds.

Question:
Is there a Borel reducibility counterpart of the Main Gap Theorem that does not need to force diamonds?

It can be studied in two ways:

- Does it holds $E^2_\omega \leqslant_B \cong^\kappa_T$ for every theory T non-classifiable under some cardinal assumptions that imply $\diamondsuit(S^\kappa_\omega)$?
- Is there a Borel reducibility counterpart of the Main Gap Theorem in another space?

Outline

The generalized Baire space

Let κ be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

The collection of Borel subsets of κ^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ.

Reductions in GBS

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

We write $E_1 \leqslant_B E_2$.

Coding structures in GBS

Fix a language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in \kappa^\kappa$ define the structure $\mathcal{A}_f$ with domain κ by: for every tuple $(a_1, a_2, \dots, a_n)$ in κ^n

$$(a_1, a_2, \dots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \dots, a_n)) > 0$$

Definition (The isomorphism relation)

Given T a first-order complete countable theory in a countable vocabulary, we say that $f, g \in \kappa^\kappa$ are $\cong_T^\kappa$ equivalent if

- $\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
  or
- $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

The Equivalence Modulo Non-stationary Ideals in GBS

We say that $f, g \in \kappa^\kappa$ are E^κ_λ equivalent if the set $\{\alpha < \kappa | f(\alpha) = g(\alpha)\}$ contains an unbounded set that is closed under λ-limits.

Theorem (Hyttinen, M.)

Suppose T is a classifiable theory and $\lambda < \kappa$ is a regular cardinal.
Then $\cong^\kappa_T \leqslant_B E^\kappa_\lambda$.

Orthogonal Chain Property (OCP)

Lemma (Hyttinen, M.)

If a theory T has the OCP, then T is not classifiable.

Theorem (Hyttinen, M.)

Suppose T is a classifiable theory, T' is an stable theory with the OCP, and κ an inaccessible cardinal. Then $\cong^{\kappa}_{T} \leqslant_B E^{\kappa}_{\omega} \leqslant_B \cong^{\kappa}_{T'}$

Strong DOP (S-DOP)

Lemma

If a theory T has the S-DOP, then T is not classifiable.

Theorem (M.)

Suppose T is a classifiable theory, T' is a superstable theory with the S-DOP, $\lambda \geq 2^\omega$, and κ an inaccessible cardinal. Then $\cong^\kappa_T \leqslant_B E^\kappa_\lambda \leqslant_B \cong^\kappa_{T'}$

Motivation

- It is consistent that there is a generalized Borel reducibility counterpart of the Main Gap Theorem in the space 2^κ.

- For κ inaccessible, the classifiable theories are at most as complex as the theories with OCP or S-DOP.

1. For which λ holds $E^\kappa_\lambda \leqslant_B E^2_\lambda$?
2. For which λ holds $E^2_\omega \leqslant_B E^2_\lambda$?

Outline

Σ_1^1-completeness

Theorem (Hyttinen, Kulikov)

If $V = L$, then E_μ^κ is Σ_1^1-complete for every $\mu < \kappa$.

Corollary

If $V = L$ and T is a theory with the OCP or the S-DOP, then $\cong_T^\kappa$ is Σ_1^1-complete.

Borel*-codes

For every regular cardinal $\gamma < \kappa$ define the following Borel*-code. Define T_γ as the tree whose elements are all the increasing elements of $\kappa^{\leqslant\gamma}$, ordered by end-extension. For every element of T_γ that is not a leaf, define

$$H_\gamma(x) = \begin{cases} \cup & \text{if } x \text{ has an immediate predecessor } x^- \text{ and } H_X(x^-) = \cap \\ \cap & \text{otherwise} \end{cases}$$

and for every leaf b define $H_\gamma(b)$ by:

$$(\eta, \xi) \in H_\gamma(b) \iff \alpha = sup(ran(b))(\eta(\alpha) = \xi(\alpha)).$$

Let us denote by $T_\gamma \restriction_\alpha$ the subtree of $T_\gamma \cap \alpha^{\leqslant\gamma}$ with

$$\{b \in T_\gamma \restriction_\alpha | b \text{ a leaf}\} = \{b \in T_\gamma | b \text{ a leaf}\} \cap \{b \in T_\gamma \cap \alpha^{\leqslant\gamma} | b \text{ a leaf}\}$$

and H^α_γ is H_γ restricted to $\{b \in T_\gamma \restriction_\alpha | b \text{ a leaf}\}$.

Borel*-reflection

Definition

For every $\gamma < \lambda < \kappa$ regular cardinals, we say that S^κ_γ Borel-reflects to S^κ_λ if the following holds for every $\eta, \xi \in \kappa^\kappa$:*

$II \uparrow B^*(T_\gamma, H_\gamma, (\eta, \xi)) \Leftrightarrow II \uparrow B^*(T_\gamma \restriction_\alpha, H^\alpha_\gamma, (\eta, \xi))$ *for λ-club many α's in S^κ_λ*

Lemma

If S^κ_γ Borel-reflects to S^κ_λ, then $E^\kappa_\gamma \leqslant_B E^\kappa_\lambda$.*

$\diamond$-reflection

Definition

Let X, Y be subsets of κ and suppose Y consists of ordinals of uncountable cofinality. We say that X $\diamond$-reflects to Y if there exists a sequence $\{D_\alpha\}_{\alpha\in Y}$ such that:

- *$D_\alpha \subset \alpha$ is stationary in α.*
- *if $Z \subset X$ is stationary, then $\{\alpha \in Y | D_\alpha = Z \cap \alpha\}$ is stationary.*

Theorem (S. Friedman, Hyttinen, Kulikov)

If X $\diamond$-reflects to Y, then $E^2_X \leqslant_B E^2_Y$.

Lemma

Suppose $\lambda^{<\lambda} = \lambda$ and $\gamma < \lambda$ regular cardinals. If S^κ_γ $\diamond$-reflects to S^κ_λ, then S^κ_γ Borel-reflects to S^κ_λ.*

Lemma

Suppose that κ is a weakly compact cardinal and that $V = L$. Then there is a forcing extension where $\lambda^{++} = \kappa$ and

1. $E^2_\lambda \leqslant_B E^2_{\lambda^+}$.
2. $E^\kappa_\lambda \leqslant_B E^\kappa_{\lambda^+}$.

Lemma

The following is consistent:
There are λ^+ many disjoint stationary subsets of $S^{\lambda^{++}}_{\lambda^+}$ such that $S^{\lambda^{++}}_{\lambda}$ $\diamond$-reflects to S_γ for every $\gamma < \lambda^+$.

Corollary

The following is consistent:

$$E^{\lambda^{++}}_{\lambda} \leqslant_B E^{2}_{\lambda^+}.$$

Questions

1. For which λ and κ holds $E^{\kappa}_{\lambda} \leqslant_B E^{2}_{\lambda}$?

2. Is it consistent that $E^{2}_{\lambda} \leqslant_B E^{2}_{\omega}$?

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