

About filter reflection

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Generalised Baire space

Suppose κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

The generalised Baire space is the space κ^κ endowed with the bounded topology, for every $\eta \in \kappa^{<\kappa}$ the following set

$$N_\eta = \{\xi \in \kappa^\kappa \mid \eta \subseteq \xi\}$$

is a basic open set.

The generalised Cantor space is the subspace 2^κ .

Reductions

For $i < 2$, let X_i be some space from the collection $\{\theta^\kappa \mid \theta \in [2, \kappa]\}$. Let R_0 and R_1 be binary relations over X_0 and X_1 , respectively.

Definition

A function $f : X_0 \rightarrow X_1$ is said to be a *reduction of R_0 to R_1* iff, for all $\eta, \xi \in X_0$,

$$\eta R_0 \xi \text{ iff } f(\eta) R_1 f(\xi).$$

The existence of a function f satisfying this is denoted by $R_0 \hookrightarrow R_1$. If f is continuous we denote it by $R_0 \hookrightarrow_c R_1$.

Stationary reflection

Let α be an ordinal of uncountable cofinality. A set $C \subseteq \alpha$ is a club if it is closed and unbounded. A set $S \subseteq \alpha$ is stationary if for all club $C \subseteq \alpha$, $C \cap S \neq \emptyset$.

Definition

Let $\alpha \in \kappa$ be an ordinal of uncountable cofinality, and a stationary $S \subseteq \kappa$, we say that S reflects at α if $S \cap \alpha$ is stationary in α

If κ is a weakly compact cardinal, every stationary subset of κ reflects at a regular cardinal $\alpha < \kappa$.

Equivalence modulo nonstationary

Definition

For every stationary set $S \subseteq \kappa$ and $\theta \in [2, \kappa]$, the equivalence relation $=_S^\theta$ over the subspace θ^κ is defined via

$\eta =_S^\theta \xi$ iff $\{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\}$ is non-stationary.

The Roll

Let us denote by $=_{\lambda}^{\theta}$ the relation $=_S^{\theta}$ when $S = \{\alpha < \kappa \mid cf(\alpha) = \lambda\}$.

Fact (Hyttinen-M)

The isomorphism relation of any classifiable theory is continuous reducible to $=_{\lambda}^{\kappa}$ for all λ .

Under some cardinal arithmetic assumptions the following can be proved:

Fact (Friedman-Hyttinen-Kulikov)

Suppose T is a non-classifiable theory. There is a regular cardinal $\lambda < \kappa$ such that $=_{\lambda}^2$ is continuous reducible the isomorphism relation of T .

Question

Is $=_{\lambda}^{\kappa}$ Continuous-reducible to $=_{\lambda}^2$, i.e. $=_{\lambda}^{\kappa} \leftrightarrow_c =_{\lambda}^2$, for all λ ?

Comparing $=_{\mathcal{S}}^{\kappa}$ and $=_{\mathcal{S}}^2$

Fact (Asperó-Hyttinen-Kulikov-M)

If every stationary subset of X reflects at stationary many $\alpha \in Y$, then $=_X^{\kappa} \hookrightarrow_c =_Y^{\kappa}$.

Fact (Friedman-Hyttinen-Kulikov)

Suppose $V = L$, and $X \subseteq \kappa$ and $Y \subseteq \text{reg}(\kappa)$ are stationary. If every stationary subset of X reflects at stationary many $\alpha \in Y$, then $=_X^2 \hookrightarrow_c =_Y^2$.

Limitations

- ▶ For all regular cardinals $\gamma \leq \lambda < \kappa$, any $X \subseteq S_\lambda^\kappa$, X does not reflect at any $\alpha \in S_\gamma^\kappa$.
- ▶ If $\kappa = \lambda^+$ and \square_λ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that Y does not reflect at any $\alpha < \kappa$.
- ▶ Usual stationary reflection requires large cardinals.

Capturing clubs

Suppose S is stationary subset of κ , and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, \mathcal{F}_α is a filter over α .

Definition

We say that $\vec{\mathcal{F}}$ captures clubs iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is non-stationary;

For any ordinal $\alpha < \kappa$ of uncountable cofinality, denote by $CUB(\alpha)$ the club filter of subsets of α . The sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S_{\omega_1}^\kappa \rangle$ define by $\mathcal{F}_\alpha = CUB(\alpha)$, capture clubs.

Filter reflection

Suppose X and S are stationary subsets of κ , and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, \mathcal{F}_α is a filter over α .

Definition

We say that X $\vec{\mathcal{F}}$ -reflects to S iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary

Definition

We say that X \mathfrak{f} -reflects to S iff there exists a sequence of filters $\vec{\mathcal{F}}$ over a stationary subset S' of S such that X $\vec{\mathcal{F}}$ -reflects to S' .

Strong filter reflection

Definition

We say that X strongly $\vec{\mathcal{F}}$ -reflects to S iff $\vec{\mathcal{F}}$ captures clubs and, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha\}$ is stationary.

We apply the same convention for X *strongly f-reflects to S*.

Properties

Fact (Monotonicity)

For stationary sets $Y \subseteq X \subseteq \kappa$ and $S \subseteq T \subseteq \kappa$. If X \mathfrak{f} -reflects to S , then Y \mathfrak{f} -reflects to T ;

Fact

Suppose X strongly \mathfrak{f} -reflects to S . If \diamond_X holds, then so does \diamond_S .

Fact

Suppose $V = L$, then for all stationary sets $X, S \subseteq \kappa$, X \mathfrak{f} -reflects to S .

Over the limits

- ▶ Vanilla stationary reflection is a special case of filter reflection.
- ▶ For all regular cardinals $\gamma \leq \lambda < \kappa$, any $X \subseteq S_\lambda^\kappa$, X does not reflect at any $\alpha \in S_\gamma^\kappa$. S_λ^κ \mathfrak{f} -reflects to S_γ^κ is consistently true.
- ▶ If $\kappa = \lambda^+$ and \square_λ holds, then for all $X \subseteq \kappa$ there is a stationary $Y \subseteq X$ such that Y does not reflect at any $\alpha < \kappa$. Filter reflection is consistent with \square_λ .
- ▶ Filter reflection does not require large cardinals.

Equivalence Modulo a Filter

Definition

Let \mathcal{F} be a filter over α . For every $\theta \in [2, \kappa]$, the equivalence modulo \mathcal{F} , $\sim_{\mathcal{F}}^{\theta}$, over θ^{α} , is defined via

$$(\eta, \xi) \in \sim_{\mathcal{F}}^{\theta} \text{ iff } \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\} \in \mathcal{F}$$

Definition

For every $\theta, \gamma \in [2, \kappa]$, $F : \theta^{\kappa} \rightarrow \gamma^{\kappa}$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_{\alpha} \mid \alpha \in S \rangle$ a sequence of filters. We say that F captures $\vec{\mathcal{F}}$ if for all $\alpha \in S$ and $\eta, \xi \in \theta^{\kappa}$

$$\eta \upharpoonright \alpha \sim_{\mathcal{F}_{\alpha}}^{\theta} \xi \upharpoonright \alpha \text{ iff } F(\eta)(\alpha) = F(\xi)(\alpha).$$

Characterization of Filter Reflection

Theorem

Let κ be such that $\kappa^{<\kappa} = \kappa$ and $X, S \subseteq \kappa$ be stationary sets. The following are equivalent:

1. X \mathfrak{f} -reflects to S .
2. There is $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ and $F : 2^\kappa \rightarrow \kappa^\kappa$ a reduction from $=_X^2$ to $=_S^\kappa$ that captures $\vec{\mathcal{F}}$.

Filtered functions

Definition

For every $\theta, \gamma \in [2, \kappa]$, $F : \theta^\kappa \rightarrow \gamma^\kappa$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ a sequence of filters. We say that F is filtered by $\vec{\mathcal{F}}$ if for all $\alpha \in S$ and $\eta, \xi \in \theta^\kappa$

$$\eta \upharpoonright \alpha \sim_{\mathcal{F}_\alpha}^\theta \xi \upharpoonright \alpha \text{ implies } F(\eta)(\alpha) = F(\xi)(\alpha).$$

Characterization Stationary Reflection

Corollary

Let κ be such that $\kappa^{<\kappa} = \kappa$ and $X, S \subseteq \kappa$ be stationary sets such that for $\alpha \in S$, $\text{cof}(\alpha) \neq \omega$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ be the sequence of club filters. Then X stationary reflects to S if and only if there is a reduction from $=^2_X$ to $=^\kappa_S$ filtered by $\vec{\mathcal{F}}$.

Characterization of Strong Filter Reflection

Theorem

Let $X, S \subseteq \kappa$ be stationary sets. The following are equivalent.

1. X strongly \mathfrak{f} -reflects to S .
2. There is a reduction $F : 2^\kappa \rightarrow 2^\kappa$ from $=_X^2$ to $=_S^2$ such that for all $\alpha \in S$ the set

$$\{\eta \upharpoonright_\alpha^{-1} [\gamma] \mid \eta \in 2^\kappa \ \& \ F(\eta)(\alpha) = \gamma\}$$

is a filter.

Characterization of Strong Filter Reflection

Proof. $2 \Rightarrow 1$

For all $\alpha \in S$ let $\mathcal{F}_\alpha = \{\eta \upharpoonright_\alpha^{-1} [\gamma] \mid \eta \in 2^\kappa \ \& \ F(\eta)(\alpha) = \gamma\}$.

- ▶ Let $\bar{1}$ and $\bar{0}$ the constant functions to 1 and 0
- ▶ $\kappa \in \mathcal{F}_\alpha$, so $F(\bar{1}) = \bar{1}$ and $F(\bar{0}) = \bar{0}$

Characterization of Strong Filter Reflection

Proof. $2 \Rightarrow 1$

For all $\alpha \in S$ let $\mathcal{F}_\alpha = \{\eta \upharpoonright_\alpha^{-1} [\gamma] \mid \eta \in 2^\kappa \text{ \& } F(\eta)(\alpha) = \gamma\}$.

- ▶ Let C be a club and η its characteristic function
- ▶ $\eta = \overset{2}{\chi} \bar{1}$, so $F(\eta) = \overset{2}{S} F(\bar{1})$
- ▶ there is a club D s.t $D \cap S = \{\alpha \in S \mid F(\eta)(\alpha) = 1\}$
- ▶ $\forall \alpha \in D \cap S, C \cap \alpha = \eta \upharpoonright_\alpha^{-1} [1] \in \mathcal{F}_\alpha$

Characterization of Strong Filter Reflection

Proof. $2 \Rightarrow 1$

For all $\alpha \in S$ let $\mathcal{F}_\alpha = \{\eta \upharpoonright_\alpha^{-1} [\gamma] \mid \eta \in 2^\kappa \text{ \& } F(\eta)(\alpha) = \gamma\}$.

- ▶ Let $Y \subseteq X$ be stationary and η its characteristic function
- ▶ $\eta \neq_X^2 \bar{0}$, so $F(\eta) \neq_S^2 F(\bar{0})$
- ▶ $\{\alpha \in S \mid F(\eta)(\alpha) \neq 0\}$ is stationary
- ▶ $Y \cap \alpha = \eta \upharpoonright_\alpha^{-1} [1] \in \mathcal{F}_\alpha$, for stationary many α 's.

Killing Filter Reflection

Theorem

There exists a cofinality-preserving forcing extension in which, for all stationary subsets X, S of κ , X does not \mathfrak{f} -reflect to S

Force a coherent regressive C -sequence, then force with $Add(\kappa, \kappa^+)$.

Forcing Filter Reflection

Theorem

For all stationary subsets X and S of κ , there exists a $<\kappa$ -closed κ^+ -cc forcing extension, in which X \mathfrak{f} -reflects to S .

Force with Sakai's forcing

Preserving clubs capturing

Definition

Let $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ be a sequence of filters that capture clubs (in V), and \mathbb{P} a forcing notion. We say that \mathbb{P} *preserves clubs capturing in $\vec{\mathcal{F}}$* if for any \mathbb{P} -generic filter G , $V[G] \models$ “ $\vec{\mathcal{H}}$ captures clubs”, where $\vec{\mathcal{H}} = \langle \mathcal{H}_\alpha \mid \alpha \in S \rangle$ and \mathcal{H}_α is the filter generated by $\check{\mathcal{F}}_\alpha$ in $V[G]$.

Lemma

Let $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ be a sequence of filters that capture clubs. If \mathbb{P} satisfies the κ -cc, then \mathbb{P} preserves clubs capturing in $\vec{\mathcal{F}}$.

$Add(\omega, \kappa)$

Theorem

Suppose X \mathfrak{f} -reflects to S holds. Then $Add(\omega, \kappa)$ forces that X \mathfrak{f} -reflects to S .

Preserving Filter Reflection

Definition

For a stationary $X \subseteq \kappa$, a notion of forcing \mathbb{Q} satisfies the X -stationary-cc if for every sequence $\langle q_\delta \mid \delta \in Y \rangle$ of conditions in \mathbb{Q} , where Y is stationary subset of X , there is a stationary $Y' \subseteq Y$ such that q_γ is compatible with q_δ for all $\gamma, \delta \in Y'$.

Definition

We say that a sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is θ -complete if the set $\{\alpha \in S \mid \mathcal{F}_\alpha \text{ is not } \theta\text{-complete}\}$ is non-stationary.

Preserving Filter Reflection

Theorem

Suppose $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a θ -complete sequence, and $X \subseteq \kappa$ is a stationary set such that X $\vec{\mathcal{F}}$ -reflects to S . Suppose \mathbb{P} is a forcing notion with θ -cc and X -stationary-cc. Then \mathbb{P} forces that X $\vec{\mathcal{H}}$ -reflects to S .

Thank you