The Borel reducibility Main Gap

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Two notions

- Model theory notion. Classification theory (Shelah 1990)

- Set theory notion. Borel reducibility (Friedman and Stanley 1989)

The spectrum fuction

Let T be a countable theory over a countable language. Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α.

Shelah's Main Gap Theorem

Theorem (Shelah 1990)

Either, for every uncountable cardinal α, $I(T,\alpha)=2^\alpha$; or $\forall\alpha>0$, $I(T,\aleph_\alpha)<\beth_{\omega_1}(|\alpha|)$.

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Descriptive Set Theory

- **1989:** Friedman and Stanley introduced the Borel reducibility between classes of countable structures.
- **1991:** Väänänen: *A Cantor-Bendixson theorem for the space* $\omega_1^{\omega_1}$.
- **2014:** Friedman-Hyttinen-Kulikov developed GDST and a systematic comparison between the Main Gap dividing lines and the complexity given by Borel reducibility.

The bounded topology

Let κ be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

The Generalised Baire spaces

The generalised Baire space is the space κ^κ endowed with the bounded topology.

The generalised Cantor space is the subspace 2^κ.

Coding structures

Let $\omega \leq \mu \leq \kappa$ be a cardinal. Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$ and a bijection π_μ between $\mu^{<\omega}$ and μ.

Definition

For every $\eta \in \kappa^\kappa$ define the structure $\mathcal{A}_{\eta\restriction\mu}$ with domain μ as follows: For every tuple $(a_1, a_2, \ldots, a_n)$ in μ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_{\eta\restriction\mu}} \Leftrightarrow \eta(\pi_\mu(m, a_1, a_2, \ldots, a_n)) > 0.$$

The isomorphism relation

Definition

Let $\omega \leq \mu \leq \kappa$ be a cardinal and T a first-order theory in a relational countable language, we say that $f, g \in \kappa^\kappa$ are $\cong^\mu_T$ equivalent if one of the following holds:

- $\mathcal{A}_{\eta\restriction\mu} \models T, \mathcal{A}_{\xi\restriction\mu} \models T, \mathcal{A}_{\eta\restriction\mu} \cong \mathcal{A}_{\xi\restriction\mu}$
- $\mathcal{A}_{\eta\restriction\mu} \not\models T, \mathcal{A}_{\xi\restriction\mu} \not\models T$

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *reducible* to E_2, if there is a function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$. We write $E_1 \hookrightarrow_r E_2$.

We can define a partial order on the set of all first-order complete countable theories

$$T \leq^\kappa T' \textit{ iff } \cong_T \hookrightarrow_C \cong_{T'}$$

Theories

- Classifiable theories are divided into:
  - shallow,
$$I(T,\aleph_\alpha) < \beth_{\omega_1}(|\alpha|);$$
  - non-shallow,
$$I(T,\alpha) = 2^\alpha.$$
- Non-classifiable theories

Question

Question: What can we say about the Borel-reducibility between those dividing lines?

Conjecture: If T is classifiable and T' is non-classifiable, then $T \leq^\kappa T'$.

Borel-reducibility Main Gap

Theorem (M.)

Let $\mathfrak{c} = 2^\omega$. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $2^\mathfrak{c} \leq \lambda = \lambda^{\omega_1}$. If T is a classifiable theory, and T' is a non-classifiable theory, then

$$T \ \leq^\kappa \ T' \text{ and } T' \ \not\leq^\kappa \ T.$$

Equivalence modulo γ cofinality

Definition

We define the equivalence relation $=^2_\gamma \subseteq 2^\kappa \times 2^\kappa$, as follows: let $S = \{\alpha < \kappa \mid cf(\alpha) = \gamma\}$,

$$\eta =^2_\gamma \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

Classifiable theories

Theorem (Hyttinen - Kulikov - M. 2017)

Assume T is a classifiable theory and let $S = \{\alpha < \kappa \mid cf(\alpha) = \gamma\}$. If $\diamondsuit_S$ holds, then $\cong_T \hookrightarrow_C =^2_\gamma$.

The idea

- Construct the reductions.
- Construct Ehrenfeucht-Mostowski models, such that

$$f \ =^2_\gamma \ g \text{ iff } \mathcal{M}^f \cong \mathcal{M}^g.$$

- Construct ordered trees, such that

$$f \ =^2_\gamma \ g \ \Leftrightarrow A_f \cong A_g \Leftrightarrow \mathcal{M}^f \cong \mathcal{M}^g.$$

Ordered trees from the linear order

- ω_1-dense,
- (κ, ω_1)-nice, $(< \kappa)$-stable,
- κ-colorable.

Hyttinen - Tuuri's order

Definition (Hyttinen - Tuuri 1991)

Let $\mathcal{R}$ be the set of functions $f : \omega \to \kappa$, for which $|\{n \in \omega \mid f(\alpha) \neq 0\}|$ is finite.
If $f, g \in \mathcal{R}$, then $f < g$ if and only if $f(n) < g(n)$, where n is the least number such that $f(n) \neq g(n)$.

Fact (Hyttinen-Tuuri 1991)

The linear order $\mathcal{R}$ is (κ, ω)-nice and $(< \kappa)$-stable.

ω_1-dense

Let $\mathcal{Q}$ be a model of *DLO* of size 2^ω, that is ω_1-dense.

Definition

Let $\kappa \times \mathcal{Q}$ be ordered by the lexicographic order, $\mathcal{I}^0$ be the set of functions $f : \omega_1 \to \kappa \times \mathcal{Q}$ such that $f(\alpha) = (f_1(\alpha), f_2(\alpha))$, for which $|\{\alpha \in \omega_1 \mid f_1(\alpha) \neq 0\}|$ is smaller than ω_1.
If $f, g \in \mathcal{I}^0$, then $f < g$ if and only if $f(\alpha) < g(\alpha)$, where α is the least number such that $f(\alpha) \neq g(\alpha)$.

κ-colorable

Definition

Let I be a linear order of size κ. We say that I is *κ-colorable* if there is a function $F : I \to \kappa$ such that for all $B \subseteq I$, $|B| < \kappa$, $b \in I \backslash B$, and $p = tp_{bs}(b, B, I)$ such that the following hold: For all $\alpha \in \kappa$,

$$|\{a \in I \mid a \models p \ \& \ F(a) = \alpha\}| = \kappa.$$

The F^φ_ω isolation

Definition

Let $\varphi(x,y) :=$ "$y > x$", we define F^φ_ω in the following way. Let $|B| < \kappa$ and $p \in S_{bs}(B)$, $(p, A) \in F^\varphi_\omega$ if and only if $A \subseteq B$, A is finite, and there is $a \in A$ such that

$$\{a > x, x = a\} \cap p \neq \emptyset \ \& \ a \models p \restriction B \backslash \{a\}.$$

F^{φ}_{ω} saturation

Definition

C is $(F^{\varphi}_{\omega}, \kappa)$-saturated if for all $B \subseteq C$ of size smaller than κ, and $p \in S_{bs}(B)$, $(p, A) \in F^{\varphi}_{\omega}$ implies that p is realized in C.

F^{φ}_{ω}-construction

Definition

A sequence $(A, (a_i, B_i)_{i<\alpha})$ is an F^{φ}_{ω}-construction over A if for all $i < \alpha$, $(tp_{bs}(a_i, A_i), B_i) \in F^{\varphi}_{\omega}$ where $A_i = A \cup \bigcup_{j<i} a_j$.
C is F^{φ}_{ω}-constructible over A if there is an F^{φ}_{ω}-construction over A such that $C = A \cup \bigcup_{j<\alpha} a_j$.

$(F^\varphi_\omega, \kappa)$-primary

Definition

C is $(F^\varphi_\omega, \kappa)$-primary over A if it is F^φ_ω-constructible over A and $(F^\varphi_\omega, \kappa)$-saturated.

Lemma (M.)

Let $\mathfrak{c} = 2^\omega$. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $2^\mathfrak{c} \leq \lambda = \lambda^{\omega_1}$. There is an $(F^\varphi_\omega, \kappa)$-primary model over $\mathcal{I}^0$ and it is an ω_1-dense, (κ, ω_1)-nice, $(< \kappa)$-stable, and κ-colorable linear order.

Non-classifiable theories

Lemma (M.)

Let κ be strongly inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. If T is a non-classifiable theory then

$$\cong^\lambda_T \hookrightarrow_C id \hookrightarrow_C \cong_T .$$

Classifiable non-shallow

Lemma (M.)

Suppose $\kappa = \lambda^+ = 2^\lambda$. *The following reduction is strict. Let* $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. *If* T_1 *is a classifiable non-shallow theory and* T_2 *is a non-classifiable theory, then*

$$\cong^{\lambda}_{T_2} \hookrightarrow_C \cong_{T_1} \hookrightarrow_C \cong_{T_2} .$$

Classifiable shallow

Lemma (M.)

Suppose $\kappa = \lambda^+ = 2^\lambda$. The following reductions are strict. Let $\kappa = \aleph_\gamma$ be such that $\beth_{\omega_1}(|\gamma|) \leq \kappa$. Suppose T_1 is a classifiable shallow theory, T_2 a classifiable non-shallow theory, and T_3 non-classifiable theory. Then

$$\cong_{T_1} \hookrightarrow_B \cong^\lambda_{T_3} \hookrightarrow_C \cong_{T_2} .$$

General reduction

Fact (Mangraviti-Motto Ros)

Let E_1 be a Borel equivalence relation with $\gamma \leq \kappa$ equivalence classes and E_2 be an equivalence relation with θ equivalence classes. If $\gamma \leq \theta$, then $E_1 \hookrightarrow_B E_2$.

Classifiable shallow

Lemma (M.)

Suppose $\kappa > 2^\omega$ and T is a countable first-order theory in a countable vocabulary (not necessarily complete) such that $\cong_T$ has $\varrho \leq \kappa$ equivalence classes. Then for all $\alpha < \kappa$

$$\cong_T \hookrightarrow_B \alpha_\varrho \text{ and } \alpha_\varrho \hookrightarrow_L \cong_T .$$

Even more, if T is not categorical then $\cong_T \not\hookrightarrow_C \alpha_\varrho$.

$\cong_T \hookrightarrow_C =^2_\mu$, $\kappa = \lambda^+$

Theory	$\lambda = \lambda^\gamma$	$\diamondsuit_\lambda$	$\mathrm{DI}^*_{S^\kappa_\gamma}(\Pi^1_1)$
Classifiable	$\omega \leq \mu \leq \gamma$	$\mu = \lambda$	$\mu = \gamma$
Non-classifiable	Indep	Indep	$\mu = \gamma$

$=^2_\mu \hookrightarrow_C \cong_T$, $\kappa = \lambda^+$

Theory	$\lambda = \lambda^\gamma$	$2^{\mathfrak{c}} \leq \lambda = \lambda^\gamma$	$2^{\mathfrak{c}} \leq \lambda = \lambda^{<\lambda}$ & $\diamondsuit_\lambda$
Stable Unsuper-stable	$\mu = \omega$	$\mu = \omega$	$\mu = \omega$
Unstable	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \lambda$
Superstable with OTOP	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \lambda$
Superstable with DOP	?	$\omega_1 \leq \mu \leq \gamma$	$\omega_1 \leq \mu \leq \lambda$

Main Gap Dichotomy

Theorem (M.)

Let κ be inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. There exists a $<\kappa$-closed κ^+-cc forcing extension in which for any countable first-order theory in a countable vocabulary (not necessarily complete), T, one of the following holds:

- $\cong_T$ *is* $\Delta^1_1(\kappa)$;
- $\cong_T$ *is* $\Sigma^1_1(\kappa)$*-complete.*

Thank you

Article at: https://arxiv.org/abs/2308.07510