Shelah's Main Gap and the generalized Borel-reducibility

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The spectrum fuction

Let T be a countable theory over a countable language. Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T,\alpha)$?

Categoricity

- **1904:** Veble introduced categorical theories.
- **1915 - 1920:** Löwenheim-Skolem Theorem.
- **1929:** Gödel's completeness theorem.
- **1954:** Łoś and Vaught introduced κ-categorical theories.
- **1965:** Morley's categoricity theorem.

Morley's conjecture

1960's: Let T be a first-order countable theory over a countable language. For all $\aleph_0 < \lambda < \kappa$,

$$I(T, \lambda) \leq I(T, \kappa).$$

1990: Shelah proved Morley's conjecture.

Shelah's Main Gap Theorem

Theorem (Shelah 1990)

Either, for every uncountable cardinal α, $I(T,\alpha)=2^\alpha$, or $\forall\alpha>0$, $I(T,\aleph_\alpha)<\beth_{\omega_1}(|\alpha|)$.

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Descriptive Set Theory

- **1989:** Friedman and Stanley introduced the Borel reducibility between classes of countable structures.
- **1990's** Descriptive set theoretical methods are used to study model theory questions (e.g. Vaught conjecture).
- **1993:** Mekler-Väänänen κ-separation theorem.
- **2007:** Laskowski Borel-completeness of ENI-DOP.
- **2014:** Friedman-Hyttinen-Kulikov GDST and classification theory.

The bounded topology

Let κ be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

The Generalised Baire spaces

The generalised Baire space is the space κ^κ endowed with the bounded topology.

The generalised Cantor space is the subspace 2^κ.

Coding structures

Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$.

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in \kappa^\kappa$ define the structure $\mathcal{A}_f$ with domain κ and for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

The isomorphism relation

Definition

Given T a first-order complete countable theory in a countable vocabulary, we say that $f, g \in \kappa^\kappa$ are $\cong_T$ equivalent if one of the following holds:

- $\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
- $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

Reductions

Let E_1 and E_2 be equivalence relations on 2^κ. We say that E_1 is *continuous reducible* to E_2, if there is a continuous function $f : 2^\kappa \to 2^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$. We write $E_1 \hookrightarrow_C E_2$.

We can define a partial order on the set of all first-order complete countable theories

$$T \leq^\kappa T' \textit{ iff } \cong_T \hookrightarrow_C \cong_{T'}$$

Non-classifiable theories

A theory T is non-classifiable if it is a countable complete theory that satisfies one of the following:

- T is unstable;
- T is stable unsuperstable;
- T is superstable with DOP;
- T is superstable with OTOP.

Consistency

Theorem (Hyttinen - Kulikov - M. 2017)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$, and $\lambda^{<\lambda} = \lambda$. There is a κ-closed κ^+-cc forcing which forces:
If T is classifiable and T' is non-classifiable, then $T \leq^\kappa T'$ and $T' \not\leq^\kappa T$.

Classifiable and shallow

Theorem (Mangraviti - Motto Ros 2020)

Let κ be such that $\kappa > 2^\omega$. If T is classifiable and shallow with depth α, then $rk_B(\cong_T) \leq 4\alpha$.

Theorem (Mangraviti - Motto Ros 2020)

Let $\kappa = \aleph_\gamma$ be such that $\kappa^{<\kappa} = \kappa$ and $\beth_{\omega_1}(|\gamma|) \leq \kappa$. Let T, T' be countable complete first-order theories, and suppose T is classifiable and shallow, while T' is not. Then

$$\cong_T \hookrightarrow_B \cong_{T'}$$

Unsuperstable theories

Theorem (Hyttinen - Kulikov - M. 2017)

Suppose $\kappa=\lambda^+$, $2^\lambda>2^\omega$, and $\lambda^\omega=\lambda$. If T is classifiable and T' is stable unsuperstable, then $T\leq^\kappa T'$ and $T'\not\leq^\kappa T$.

Theorem (M. 2022)

Suppose $\kappa=\lambda^+=2^\lambda$ and $\lambda^\omega=\lambda$. If T is a classifiable theory, and T' is an unsuperstable theory, then $T\leq^\kappa T'$ and $T'\not\leq^\kappa T$.

Equivalence modulo γ cofinality

Definition

We define the equivalence relation $=^2_\gamma \subseteq 2^\kappa \times 2^\kappa$, as follows: let $S = \{\alpha < \kappa \mid cf(\alpha) = \gamma\}$,

$$\eta =^2_\gamma \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

Classifiable theories

Theorem (Hyttinen - Kulikov - M. 2017)

Assume T is a classifiable theory. If $\diamondsuit_S$ holds, then $\cong_T \hookrightarrow_C =^2_\gamma$.

Main result

Theorem (M. 2023)

Let $\mathfrak{c} = 2^\omega$. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $2^\mathfrak{c} \leq \lambda = \lambda^{\omega_1}$. If T is a classifiable theory, and T' is a non-classifiable theory, then there is $\gamma < \kappa$ such that

$$\cong_T \hookrightarrow_C =^2_\gamma \hookrightarrow_C \cong_{T'} \text{ and } =^2_\gamma \not\hookrightarrow_B \cong_T .$$

In particular

$$T \ \leq^\kappa \ T' \text{ and } T' \ \not\leq^\kappa \ T.$$

Blue print of the proof

- Construct an ε-dense, $(\kappa, bs, bs, \varepsilon)$-nice, $(<\kappa, bs)$-stable, and κ-colorable linear order.
- Construct ordered trees from the linear order.
- Construct skeletons from ordered trees, to construct Ehrenfeucht-Mostowski models.
- Contruct the reductions.

ε-dense

Definition

Let I be a linear order of size κ and ε a regular cardinal smaller than κ. We say that I is *ε-dense* if the following holds.

If $A, B \subseteq I$ are subsets of size less than ε such that for all $a \in A$ and $b \in B$, $a < b$, then there is $c \in I$, such that for all $a \in A$ and $b \in B$, $a < c < b$.

κ-representation

Definition

Let A be an arbitrary set of size κ. The sequence $\mathbb{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ is a *κ-representation* of A, if $\langle A_\alpha \mid \alpha < \kappa \rangle$ is an increasing continuous sequence of subsets of A, for all $\alpha < \kappa$, $|A_\alpha| < \kappa$, and $\bigcup_{\alpha<\kappa} A_\alpha = A$.

$(\kappa, bs, bs, \varepsilon)$-nice

Definition

Let $\varepsilon < \kappa$ be a regular cardinal, A be a linear order of size κ and $\langle A_\alpha \mid \alpha < \kappa \rangle$ a κ-representation. Then A is $(\kappa, bs, bs, \varepsilon)$-*nice* if there is a club $C \subseteq \kappa$, such that for all limit $\delta \in C$ with $cf(\delta) \geq \varepsilon$, for all $x \in A$ there is $\beta < \delta$ such that one of the following holds:

- $\forall \sigma \in A_\delta [\sigma \geq x \Rightarrow \exists \sigma' \in A_\beta \ (\sigma \geq \sigma' \geq x)]$
- $\forall \sigma \in A_\delta [\sigma \leq x \Rightarrow \exists \sigma' \in A_\beta \ (\sigma \leq \sigma' \leq x)]$

$(<\kappa, bs)$-stable

Definition

A linear order I is $(<\kappa, bs)$-*stable* if for every $B \subseteq I$ of size smaller than κ,

$$\kappa > |\{tp_{bs}(a, B, I) \mid a \in I\}|.$$

κ-colorable

Definition

Let I be a linear order of size κ. We say that I is *κ-colorable* if there is a function $F : I \to \kappa$ such that for all $B \subseteq I$, $|B| < \kappa$, $b \in I \backslash B$, and $p = tp_{bs}(b, B, I)$ such that the following hold: For all $\alpha \in \kappa$,

$$|\{a \in I \mid a \models p \ \& \ F(a) = \alpha\}| = \kappa.$$

The order

Let $\theta < \kappa$ be the smallest cardinal such that there is a ε-dense model of *DLO* of size θ.

Theorem

Suppose κ is inaccessible, or $\kappa = \lambda^+$, $2^\theta \leq \lambda = \lambda^{<\varepsilon}$. There is a ε-dense, $(\kappa, bs, bs, \varepsilon)$-nice, $(<\kappa, bs)$-stable, and κ-colorable linear order.

κ^+, $(\gamma+2)$-tree*

Let $\gamma<\kappa$ be a regular cardinal. A κ^+, $(\gamma+2)$-tree* t is a tree with the following properties:

- t has a unique root.
- Every element of t has less than κ^+ immediate successors.
- All the branches of t have order type γ or $\gamma+1$.
- Every chain of length less than γ has a unique limit.

Ordered trees

Definition

Let $\gamma < \kappa$ be a regular cardinal and I a linear order. $(A, \prec, <)$ is an ordered tree if the following holds:

- $(A, \prec)$ is a κ^+, $(\gamma + 2)$-tree*.
- for all $x \in A$, $(succ(x), <)$ is isomorphic to I.

Isomorphism of trees

Theorem (M. 2023)

Suppose $\gamma < \kappa$ is such that for all $\epsilon < \kappa$, $\epsilon^\gamma < \kappa$, and there is a κ-colorable linear order I. For all $f \in 2^\kappa$ there is an ordered tree A_f such that for all $f, g \in 2^\kappa$,

$$f \ =^2_\gamma \ g \ \Leftrightarrow A_f \cong A_g.$$

Skeletons

Definition

For every $f \in 2^\kappa$ let us define the order $K(f)$ by:

I. $dom\ K(f) = (dom\ A_f \times \{0\}) \cup (dom\ A_f \times \{1\})$.

II. For all $\eta \in A_f$, $(\eta, 0) <_{K(f)} (\eta, 1)$.

III. If $\eta, \xi \in A_f$, then $\eta \prec \xi$ if and only if

$$(\eta, 0) <_{K(f)} (\xi, 0) <_{K(f)} (\xi, 1) <_{K(f)} (\eta, 1).$$

IV. If $\eta, \xi \in A_f$, then $\eta < \xi$ if and only if $(\eta, 1) <_{K(f)} (\xi, 0)$.

The models

Suppose κ is inaccessible, or $\kappa = \lambda^+$, $2^\theta \leq \lambda = \lambda^{<\varepsilon}$. Let $\gamma < \kappa$ be such that for all $\epsilon < \kappa$, $\epsilon^\gamma < \kappa$.

Lemma (M. 2023)

Suppose T is superstable with DOP in a countable relational vocabulary τ. Let τ^1 be a Skolemization of τ, and T^1 be a complete theory in τ^1 extending T and with Skolem-functions in τ. Then for every $f \in 2^\kappa$ there is $\mathcal{M}_1^f \models T^1$ with the following properties.

The models

Lemma (M. 2023)

1. *There is a map* $\mathcal{H} : A_f \to (\text{dom } \mathcal{M}_1^f)^n$ *for some* $n < \omega$, $\eta \mapsto a_\eta$, *such that* $\mathcal{M}_1^f$ *is the Skolem hull of* $\{a_\eta \mid \eta \in A_f\}$. *Let us denote* $\{a_\eta \mid \eta \in A_f\}$ *by* $Sk(\mathcal{M}_1^f)$.
2. $\mathcal{M}^f = \mathcal{M}_1^f \restriction \tau$ *is a model of* T.
3. $Sk(\mathcal{M}_1^f)$ *is indiscernible in* $\mathcal{M}_1^f$ *relative to* $L_{\omega_1\omega_1}$.
4. *There is a formula* $\varphi \in L_{\omega_1\omega_1}(\tau)$ *such that for all* $\eta, \nu \in A_f$ *and* $m < \gamma$, *if* $A_f \models P_m(\eta) \wedge P_\gamma(\nu)$, *then* $\mathcal{M}^f \models \varphi(a_\nu, a_\eta)$ *if and only if* $A_f \models \eta \prec \nu$.

The isomorphism theorem

Theorem (M. 2023)

Suppose T is a non-classifiable first order theory in a countable relational vocabulary τ.

1. *If T is unstable or superstable with OTOP, $\omega \leq \gamma < \kappa$ is such that for all $\alpha < \kappa$, $\alpha^\gamma < \kappa$, then for all $f, g \in 2^\kappa$*
$$f =^2_\gamma g \text{ iff } \mathcal{M}^f \cong \mathcal{M}^g.$$
2. *If T is superstable with DOP, κ is inaccessible or $\kappa = \lambda^+$ and $2^{\mathfrak{c}} \leq \lambda$, and $\omega_1 \leq \gamma < \kappa$ is such that for all $\alpha < \kappa$, $\alpha^\gamma < \kappa$, then for all $f, g \in 2^\kappa$,*
$$f =^2_\gamma g \text{ iff } \mathcal{M}^f \cong \mathcal{M}^g.$$

The reductions

Theorem (M. 2023)

Let κ be inaccessible or $\kappa = \lambda^+ = 2^\lambda$. Suppose T is a non-classifiable theory.

1. *If T is stable unsuperstable, then let $\theta = \gamma = \omega$.*
2. *If T is unstable, or superstable with OTOP, then let $\theta = \omega$ and $\omega \leq \gamma < \kappa$.*
3. *If T is superstable with DOP, then let $\theta = 2^\omega = \mathfrak{c}$ and $\omega_1 \leq \gamma < \kappa$.*

If θ, γ, and κ satisfy that $\forall \alpha < \kappa$, $\alpha^\gamma < \kappa$, and $(2^\theta)^+ \leq \kappa$, then

$$=^2_\gamma \hookrightarrow_C \cong_T .$$

$\cong_T \hookrightarrow_C =^2_\mu$, $\kappa = \lambda^+$

Theory	$\lambda = \lambda^\gamma$	$\diamondsuit_\lambda$	$\mathrm{DI}^*_{S^\kappa_\gamma}(\Pi^1_1)$
Classifiable	$\omega \leq \mu \leq \gamma$	$\mu = \lambda$	$\mu = \gamma$
Non-classifiable	Indep	Indep	$\mu = \gamma$

$=^2_\mu \hookrightarrow_C \cong_T$, $\kappa = \lambda^+$

Theory	$\lambda = \lambda^\gamma$	$2^{\mathfrak{c}} \leq \lambda = \lambda^\gamma$	$2^{\mathfrak{c}} \leq \lambda = \lambda^{<\lambda}$ & $\diamondsuit_\lambda$
Stable Unsuperstable	$\mu = \omega$	$\mu = \omega$	$\mu = \omega$
Unstable	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \lambda$
Superstable with OTOP	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \gamma$	$\omega \leq \mu \leq \lambda$
Superstable with DOP	?	$\omega_1 \leq \mu \leq \gamma$	$\omega_1 \leq \mu \leq \lambda$

A bigger Gap

Theorem (M. 2023)

Suppose κ is inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^\mathfrak{c} \leq \lambda = \lambda^{\omega_1}$ and $\lambda^{<\lambda} = \lambda$. There exists a cofinality-preserving forcing extension in which the following holds:
If T_1 is classifiable and T_2 is not. Then there is a regular cardinal $\gamma < \kappa$ such that, if $X, Y \subseteq S^\kappa_\gamma$ are stationary and disjoint, then $=^2_X$ and $=^2_Y$ are strictly in between $\cong_{T_1}$ and $\cong_{T_2}$.

Main Gap Dichotomy

Theorem (M. 2023)

Let κ be inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. There exists a $<\kappa$-closed κ^+-cc forcing extension in which for any countable first-order theory in a countable vocabulary (not necessarily complete), T, one of the following holds:

- $\cong_T$ *is* Δ^1_1;
- $\cong_T$ *is* Σ^1_1*-complete.*

On Morley's Conjecture

Lemma (M. 2023)

Let κ be strongly inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^{\mathfrak{c}} \leq \lambda = \lambda^{<\omega_1}$. For all cardinals $\aleph_0 < \mu < \delta < \kappa$, if T is a non-classifiable theory then

$$\cong^\mu_T \hookrightarrow_C \cong^\delta_T \hookrightarrow_C id \hookrightarrow_C \cong_T .$$

The reductions also hold for T a classifiable non-shallow theory, when κ is a strongly inaccessible cardinal.

Thank you

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