Isolation notions and construction of models

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This notes are based on a series of talks given at the Research Model Theory seminar of University of Vienna. This notes are intended to be as close as possible to the transcripts of those seminar session. Due to the nature of the seminar and the questions from the audience, the proofs are presented with more detail in these notes.

Contents

1 Preliminaries

In order to simplify the notation, we will use the monster model technique, i.e. we work inside $\mathbb{M}$, where $\mathbb{M} \models T$ is a saturated model of size $\bar{\kappa}$. $\bar{\kappa}$ is larger than the cardinality of any object that we come across. By a model, we mean an elementary submodel of $\mathbb{M}$ of size smaller than $\bar{\kappa}$. By $a \in A$ we mean $a \in A^{length(a)}$. By *"a model"* we mean an elementary submodel of $\mathbb{M}$ of size smaller than $\bar{\kappa}$. We will mainly work with stable theories (Definition 2.8).

Notation. We denote $S^m(A)$ the set of all consistent types over A in m variables (modulo change of variables). $S(A) = \cup_{m<\omega} S^m(A)$ and by $t(a, A)$ we mean the complete type of A (in $\mathbb{M}$).

For mention some examples and properties, we will need the forking notion. We will follow the definition of forking from [2], it is equivalent to the definition from [12]

Definition 1.1. *For every finite set Δ of formulas, we define $R_\Delta(p, \omega)$, for all types p, in the following way.*

1. *$R_\Delta(p, \omega) \geq 0$ if p is consistent.*
2. *$R_\Delta(p, \omega) \geq \alpha + 1$ if for all finite $q \subseteq p$ and $n < \omega$, there are Δ-types $\{q_i\}_{i<n}$, such that:*
   (a) *for all $i < j \leq n$ there are $\varphi(x, y) \in \Delta$ and b such that $\varphi(x, b) \in q_i$ and $\neg\varphi(x, b) \in q_j$ or vice versa,*
   (b) *for all $i < n$, $R_\Delta(q \cup q_i, \omega) \geq \alpha$.*
3. *If α is limit, then $R_\Delta(p, \omega) \geq \alpha$ if $R_\Delta(p, \omega) \geq \beta$ holds for all $\beta < \alpha$.*

We say that $R_\Delta(p, \omega) = \alpha$ if α is the least ordinal such that $R_\Delta(p, \omega) \ngeq \alpha$. If such α does not exists, then we say $R_\Delta(p, \omega) = \infty$. We write $R_\Delta(p, \omega) = -1$ if p is not consistent.

Definition 1.2. *We say that a consistent formula $\varphi(x, a)$, $a \in \mathbb{M}$, forks over A if for all $p = p(x) \in S(A)$ the following holds:*

If $p \cup \{\varphi(x, a)\}$ is consistent, then there is a finite Δ such that for all finite $\Delta' \supseteq \Delta$, $R_{\Delta'}(p \cup \{\varphi(x, a)\}, \omega) < R_{\Delta'}(p, \omega)$.

Definition 1.3. *We say that p forks over A if there is a finite $q \subseteq p$ such that $\wedge q$ forks over A*

Definition 1.4. *We write $a \downarrow_A B$ if $t(a, A \cup B)$ does not fork over A.*

Lemma 1.5 (Properties of forking, [2], [12])**.** *Let $A \subseteq B \subseteq C \subseteq D$, a and b be arbitrary.*

1. *$a \downarrow_A A$.*
2. *If $a \downarrow_A D$, then $a \downarrow_B C$.*
3. *If $a \not\downarrow_A B$, then there is $c \in B$ such that $a \not\downarrow_A c$.*
4. *If $a \downarrow_A b$, then $b \downarrow_A a$*
5. *$a \downarrow_A C$ if and only if $a \downarrow_A B$ and $a \downarrow_B C$.*
6. *If $t(a, A)$ is algebraic, then $a \downarrow_A B$ for all B.*
7. *If $t(a, B)$ is algebraic and $t(a, A)$ is not, then $a \not\downarrow_A B$.*

2 Motivation: Classifiaction theory and GDST

The aim of this section is to explain why the isolation notions and primary models are important when study the isomorphism relation of countable theories in Generalized Descriptive Set Theory (GDST). Some of the notions defined in this section require definitions from the following sections for a full understanding. The idea of presenting this notions without definition is to provide a simplify picture of the notions that we will deal with in the following sections.

We will work under the general assumption that κ is a regular uncountable cardinal that satisfies $\kappa = \kappa^{<\kappa}$. We will work only with first-order countable complete theories on a countable language, unless something else is stated.

Definition 2.1 (The Generalized Baire space)**.** *Let κ be an uncountable cardinal. The generalized Baire space is the set κ^κ endowed with the following topology. For every $\eta \in \kappa^{<\kappa}$, define the following basic open set*

$$N_\eta = \{f \in \kappa^\kappa \mid \eta \subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

Definition 2.2 (The Generalized Cantor space). *Let κ be an uncountable cardinal. The generalized Cantor space is the set 2^κ endowed with the following topology. For every $\eta \in 2^{<\kappa}$, define the following basic open set*

$$N_\eta = \{f \in 2^\kappa \mid \eta \subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

Let us fix a bijection $\pi : \kappa^{<\omega} \to \kappa$ and a countable relational language $\mathcal{L} = \{P_m \mid m \in \omega\}$.

Definition 2.3. *For every $\eta \in \kappa^\kappa$ define the structure $\mathcal{A}_\eta$ with domain κ as follows.*
For every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \Leftrightarrow \text{ the arity of } P_m \text{ is } n \text{ and } \eta(\pi(m, a_1, a_2, \ldots, a_n)) > 0.$$

Definition 2.4. *For every $\eta \in 2^\kappa$ define the structure $\mathcal{A}_\eta$ with domain κ as follows.*
For every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \Leftrightarrow \text{ the arity of } P_m \text{ is } n \text{ and } \eta(\pi(m, a_1, a_2, \ldots, a_n)) = 1.$$

Notice that the previous method can also be used to encode structures with domain α, into functions α^α, $\mathcal{A}_\alpha$. The structure $\mathcal{A}_\eta \restriction \alpha$ is not necessary coded by the function $\eta \restriction \alpha$.

Exercise 2.1. *There is a club C_π such that for all $\alpha \in C_\pi$, $\mathcal{A}_\eta \restriction \alpha = \mathcal{A}_{\eta\restriction\alpha}$*

With the structures coded by the elements of 2^κ and κ^κ, it is easy to define the isomorphism relation of structures of size κ in both spaces.

Definition 2.5 (The isomorphism relation). *Assume T is a complete first order theory in a countable vocabulary. We define $\cong^\kappa_T$ as the relation*

$$\{(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \mid (\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T)\}.$$

Definition 2.6. *Assume T is a complete first order theory in a countable vocabulary. We define $\cong^2_T$ as the relation*

$$\{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T)\}.$$

The collection of κ-Borel subsets of κ^κ is the smallest set that contains the basic open sets and is closed under union and intersection both of length κ. A κ-Borel set is any set of this collection.

A function $f\colon \kappa^\kappa \to \kappa^\kappa$ is κ-Borel, if for every open set $A \subseteq \kappa^\kappa$ the inverse image $f^{-1}[A]$ is a κ-Borel subset of X. Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *κ-Borel reducible to* E_2 if there is a κ-Borel function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies

$$(\eta, \xi) \in E_1 \iff (f(\eta), f(\xi)) \in E_2.$$

We call f a reduction of E_1 to E_2 and we denote this by $E_1 \hookrightarrow_B E_2$. In the case f is continuous, we say that E_1 is continuously reducible to E_2 and we denote it by $E_1 \hookrightarrow_c E_2$.

Notice that $\cong^\kappa_T \hookrightarrow_c \cong^2_T$ holds for every theory T. From now on let us denote by $\cong_t$ both notions $\cong^\kappa_T$ and $\cong^2_T$.

Question 2.7. *Under which assumptions on theories T_1 and T_2 the following holds*

$$\cong_{T_1} \hookrightarrow_c \cong_{T_2},$$

or even

$$\cong_{T_1} \hookrightarrow_B \cong_{T_2}?$$

2.1 Classifiable and non-classifiable theories

Shelah's Main Gap Theorem gives us a notion of complexity, a theory is more complex if it has more models. Thus, it gives us an idea on how the Borel-reducibility of the isomorphism relation of theories may behave. Let us introduce the require notions to state Shelah's Main Gap Theorem.

Definition 2.8.

- *We say that T is ξ-stable if for any set A, $|A| \leq \xi$, $|S(A)| \leq \xi$.*
- *We say that T is stable if there is an infinite ξ, such that T is ξ-stable.*
- *We say that T is unstable if there is no infinite ξ, such that T is ξ-stable.*

- *We say that T is superstable is there is an infinite ξ such that for all $\xi' > \xi$, T is ξ'-stable.*

Definition 2.9 (OTOP). *A theory T has the omitting type order property (OTOP) if there is a sequence $(\varphi_m)_{m<\omega}$ of first order formulas such that for every linear order l there is a model $\mathcal{M}$ and n-tuples $a_t (t \in l)$ of members of $\mathcal{M}$, $n < \omega$, such that $s < t$ if and only if there is a k-tuple c of members of $\mathcal{M}$, $k < \omega$, such that for every $m < \omega$,*

$$\mathcal{M} \models \varphi_m(c, a_s, a_t).$$

Definition 2.10 (DOP). *A theory T has the dimensional order property (DOP) if there are F^a_ω-saturated models $(M_i)_{i<3}$, $M_0 \subseteq M_1 \cap M_2$, $M_1 \downarrow_{M_0} M_2$, and the F^a_ω-prime model over $M_1 \cup M_2$ is not F^a_ω-minimal over $M_1 \cup M_2$.*

Definition 2.11.
- *We say that T is classifiable is T is superstable without DOP and without OTOP.*
- *We say that T is non-classifiable if it satisfies one of the following:*
  1. *T is stable unsuperstable;*
  2. *T is superstable with DOP;*
  3. *T is superstable with OTOP;*
  4. *T is unstable.*

Theorem 2.12 ([12] Main Gap Theorem). *For every T first order complete theory over a countable vocabulary. Let $I(T, \alpha)$ denote the number of non-isomorphic models of T with cardinality α. One of the following holds:*

1. *If T is shallow superstable without DOP and without OTOP, then $\forall \alpha > 0$ $I(T, \aleph_\alpha) \leq \beth_{\omega_1}(| \alpha |)$.*
2. *If T is not superstable, or superstable and deep or with DOP or with OTOP, then for every uncountable cardinal α, $I(T, \alpha) = 2^\alpha$.*

Question 2.13. *Let T_1 be a classifiable theory and T_2 be a non-classifiable theory. Is $\cong_{T_1}$ Borel-reducible (or continuous) to $\cong_{T_2}$, i.e. $\cong_{T_1} \hookrightarrow_B \cong_{T_2}$?*

2.2 The successor case

The equivalent modulo non-stationary S ($=^\kappa_S$) has been very important when study Question 2.13.

Definition 2.14. *Given $S \subseteq \kappa$ and $\beta \leq \kappa$, we define the equivalence relation $=^\beta_S \subseteq \beta^\kappa \times \beta^\kappa$, as follows*

$$\eta =^\beta_S \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

Let μ be a regular cardinal. We will denote by $=^\beta_\mu$ the relation $=^\beta_S$ when $S = \{\alpha < \kappa \mid cf(\alpha) = \mu\}$. Notice that $\eta =^\beta_\mu \xi$ holds if and only if $\{\alpha < \kappa \mid cf(\alpha) = \mu \;\&\; \eta(\alpha) = \xi(\alpha)\}$ contains an unbounded subset closed under μ-sequences.

The following is the usual Ehrenfeucht-Fraïssé game but coded in a particular way for our purposes.

Definition 2.15. *(Ehrenfeucht-Fraïssé game) Fix $\{X_\gamma\}_{\gamma<\kappa}$ an enumeration of the elements of $\mathcal{P}_\kappa(\kappa)$ and $\{f_\gamma\}_{\gamma<\kappa}$ an enumeration of all the functions with domain in $\mathcal{P}_\kappa(\kappa)$ and range in $\mathcal{P}_\kappa(\kappa)$. For every pair of structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ and $\alpha < \kappa$, the $EF^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ is a game played by the players* **I** *and* **II** *as follows.*
In the n-th move, first **I** *choose an ordinal $\beta_n < \alpha$ such that $X_{\beta_n} \subset \alpha$, $X_{\beta_{n-1}} \subseteq X_{\beta_n}$, and then* **II** *an ordinal $\theta_n < \alpha$ such that $dom(f_{\theta_n}), rang(f_{\theta_n}) \subset \alpha$, $X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}} \subseteq f_{\theta_n}$ (if $n = 0$ then $X_{\beta_{n-1}} = \emptyset$ and $f_{\theta_{n-1}} = \emptyset$). The game finishes after ω moves. The player* **II** *wins if $\cup_{i<\omega} f_{\theta_i} : A\restriction_\alpha \to B\restriction_\alpha$ is a partial isomorphism, otherwise the player* **I** *wins.*

We write **I** $\uparrow$ $\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ if **I** has a winning strategy in the game $\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$. We write **II** $\uparrow$ $\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ if **II** has a winning strategy.

Lemma 2.16 ([4], Lemma 2.4). *If $\mathcal{A}$ and $\mathcal{B}$ are structures with domain κ, then the following hold:*

- **II** $\uparrow$ *$EF^\kappa_\omega(\mathcal{A}\restriction\kappa, \mathcal{B}\restriction\kappa) \iff$* **II** $\uparrow$ *$EF^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ for club-many α.*
- **I** $\uparrow$ *$EF^\kappa_\omega(\mathcal{A}\restriction\kappa, \mathcal{B}\restriction\kappa) \iff$* **I** $\uparrow$ *$EF^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ for club-many α.*

The reason to introduce these games is that we can characterize classifiable theories with these games.

Theorem 2.17 ([12], XIII Theorem 1.4). *If T is a classifiable theory, then every two models of T that are $L_{\infty,\kappa}$-equivalent are isomorphic.*

Theorem 2.18 ([1], Theorem 10). *$L_{\infty,\kappa}$-equivalence is equivalent to EF^κ_ω-equivalence.*

From these two theorems we know that if T is a classifiable theory, then for any $\mathcal{A}$ and $\mathcal{B}$ models of T with domain κ,

$$\mathbf{II}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A},\mathcal{B}) \iff \mathcal{A}\cong\mathcal{B}$$

$$\mathbf{I}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A},\mathcal{B}) \iff \mathcal{A}\ncong\mathcal{B}.$$

From the previous Lemma we know the following two hold for any $\mathcal{A}$ and $\mathcal{B}$ models of a classifiable theory (with domain κ):

- $\mathcal{A}\cong\mathcal{B} \iff \mathbf{II}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α.
- $\mathcal{A}\ncong\mathcal{B} \iff \mathbf{I}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α.

Lemma 2.19 ([3], Lemma 2). *Let $\mu<\kappa$ be a regular cardinal and $S^\kappa_\mu=\{\alpha<\kappa\mid cf(\alpha)=\mu\}$. Assume T is a classifiable theory and $\mu<\kappa$ is a regular cardinal. If $\diamondsuit_\kappa(S^\kappa_\mu)$ holds then $\cong_T$ is continuously reducible to $=^2_\mu$.*

By using Ehrenfeucht-Mostowski models, it is possible to construct for each $f\in 2^\kappa$ a model $\mathcal{A}^f$ such that the following hold.

Lemma 2.20 ([10], Lemma 4.28). *Suppose $\kappa=\lambda^+=2^\lambda$ and $\lambda^\omega=\lambda$. If T is a countable complete unsuperstable theory over a countable vocabulary, then for all $f,g\in 2^\kappa$, $f\ =^2_\omega\ g$ if and only if $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic. Even more, $=^2_\omega\ \hookrightarrow_c\ \cong_T$.*

Theorem 2.21 ([10], Corollary 4.12). *Suppose $\kappa=\lambda^+=2^\lambda$ and $\lambda^\omega=\lambda$. If T_1 is a countable complete classifiable theory, and T_2 is a countable complete unsuperstable theory, then $\cong_{T_1}\ \hookrightarrow_c\ \cong_{T_2}$ and*

$$\cong_{T_2}\ \not\hookrightarrow_c\ \cong_{T_1}.$$

Theorem 2.22 ([11], Theorem 5.5). *Suppose $\kappa=\lambda^+=2^\lambda$, $2^\omega=\mathfrak{c}$, and $2^{\mathfrak{c}}\geq\lambda=\lambda^{\omega_1}$. Let T be a countable complete classifiable theory and T_2 be a countable complete non-classifiable theory. If T_2 is superstable with OTOP, or superstable with DOP, then*

$$\cong_{T_1}\ \hookrightarrow_c\ \cong_{T_2}\ \text{ and }\ \cong_{T_2}\not\hookrightarrow_B\ \cong_{T_1}.$$

2.3 Inaccessible case

The previous results depend highly on $\diamondsuit_\kappa(S^\kappa_\omega)$ and $\diamondsuit_\kappa(S^\kappa_{\omega_1})$. This diamond principle holds under the assumptions $\kappa=\lambda^+=2^\lambda$ and $\lambda=\lambda^{\omega_1}$. For the case κ inaccessible we will need to change the approach. Fortunately there is a version of Lemma 2.19 for the generalized Baire space, which is a theorem of ZFC.

Lemma 2.23 ([4], Theorem 2.8). *Assume T is a classifiable theory and $\mu<\kappa$ is a regular cardinal. Then $\cong_T$ is continuously reducible to $=^\kappa_\mu$.*

In the inaccessible case, the construction of the models is done by using primary models. The models $\mathcal{A}^f$ are constructed such that there is $\mu<\kappa$ such that for all $f,g\in\kappa^\kappa$, $f=^\kappa_\mu g$ holds if and only if $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic.

To show the construction, we will need to introduce the basics of isolation notions and constructible sets.

3 Isolation notions

In this section we will follow Hyttinen notes [2] and the fourth chapter of [12]. We will omit some proofs, these ones can be found in [2] and [12].

3.1 Axioms and Examples

Let λ be an infinite cardinal and P_λ the class of pairs (p,A) such that $|A|<\lambda$ and for some $B\supseteq A$, $p\in S(B)$. Let $F_\lambda\subseteq P_\lambda$ be such that satisfies the following axioms:

I. F_λ is closed under the change of variables. If $p\in S(B)$, $A\subseteq B$ and

$$q=\{\varphi(y_0,\dots,y_m;\bar a)\mid \varphi(x_{\sigma(0)},\dots,x_{\sigma(m)};\bar a)\in p\}$$

(where σ is a permutation of $\{0,\dots,m\}$), then $(p,A)\in F_\lambda$ if and only if $(q,A)\in F_\lambda$.

II. F_λ is closed under automorphisms. For all automorphism f, $(p, A) \in F_\lambda$ holds if and only if $(f(p), f[A]) \in F_\lambda$.

III. If $a \in A \subseteq B$ and $|A| < \lambda$, then $(t(a, B), A) \in F_\lambda$.

IV. If $A \subseteq B \subseteq C \subseteq dom(p)$, $|B| < \lambda$ and $(p, A) \in F_\lambda$, then $(p \upharpoonright C, B) \in F_\lambda$.

V. If $(t(a \cup b, B), A) \in F_\lambda$, then $(t(a, B), A) \in F_\lambda$.

VI. If $|C| < \lambda$ and $(t(a \cup C, B), A) \in F_\lambda$, then $(t(a, B \cup C), A \cup C) \in F_\lambda$.

VII. If $A, B \subseteq C$, $(t(b, C \cup a), B) \in F_\lambda$ and $(t(a, C), A) \in F_\lambda$, then $(t(a, C \cup b), A) \in F_\lambda$.

VIII. If $A \subseteq B$, $(t(a, B \cup C), A \cup C) \in F_\lambda$ and $(t(C, B), A) \in F_\lambda$, then $(t(a \cup C, B), A) \in F_\lambda$.

IX. If $\langle B_i \mid i < \delta \rangle$ is an increasing sequence of sets, $p \in S(\bigcup_{i<\delta} B_i)$ and for all $i < \delta$, $(p \upharpoonright B_i, A) \in F_\lambda$, then $(p, A) \in F_\lambda$.

X. If $(p, A) \in F_\lambda$ and $dom(p) \subseteq B$, then there are $A' \subseteq B$ and $q \in S(B)$ such that $p \subseteq q$ and $(q, A') \in F_\lambda$.

We write $(t(C, B), A) \in F_\lambda$ if for every $c \in C$, $(t(c, B), A) \in F_\lambda$.

Definition 3.1. *Let us define F^t_λ to be the set of all pairs $(p, A) \in P_\lambda$ that satisfies: For all $(p, A) \in F^t_\lambda$, there is $q \subseteq p \upharpoonright A$ such that $|q| < \lambda$ and $q \vdash p$.*

Lemma 3.2 ([2], [12])**.** *F^t_λ satisfies axioms I to IX. If $\lambda > |T|$ and T is λ-stable, then it also satisfies axiom X.*

Definition 3.3. *Let us define F^s_λ to be the set of all pairs $(p, A) \in P_\lambda$ such that $p \upharpoonright A \vdash p$.*

Lemma 3.4 ([2], [12])**.** *F^s_λ satisfies axioms I to IX. If $\lambda > |T|$ and T is λ-stable, then it also satisfies axiom X.*

Proof. Axioms I to IV are easy to show.

V. Let a, b, A, B be such that $(t(a \cup b, B), A) \in F_\lambda$. By the definition of F^s_λ, we know that $t(a \cup b, A) \vdash t(a \cup b, B)$. So $t(a, A) \vdash t(a, B)$.

VI. Let a, A, B, C be such that $(t(a \cup C, B), A) \in F_\lambda$. Let b be such that $b \models t(a, A \cup C)$. Therefore $b^\frown C \models t(a \cup C, A)$. Since $(t(a \cup C, B), A) \in F_\lambda$, $b^\frown C \models t(a \cup C, B)$. We conclude that $b \models t(a, B \cup C)$.

VII. Let us show that if $A, B \subseteq C$, $(q_1, B) \in F^s_\lambda$ where $q_1 = t(b, C \cup a)$ and $(p, A) \in F^s_\lambda$ where $p = t(a, C)$, then $(p_1, B) \in F^s_\lambda$, where $p_1 = t(a, C \cup b)$.

Let $q = q_1 \upharpoonright A$. Let us first show that if a_1 and b_1 are such that $a_1 \models p$ and $b_1 \models q$, then $a_1^\frown b_1 \models t(a^\frown b, C)$. Since $t(a_1, C) = t(a, C)$, there is an elementary map f such that $f \upharpoonright C = id_C$ and $f(a) = a_1$. On the other hand $B \subseteq C$, and $(q, B) \in F^s_\lambda$. So $t(b, C) \vdash t(b, C \cup a)$ and $t(b, C) \vdash t(b, C \cup a_1)$. Therefore $t(a^\frown b, C) = t(a_1^\frown b_1, C)$.

So $t(a_1, C \cup b) = t(a, C \cup b)$. Thus $p \vdash p_1$. Finally $(t(a, C), A) \in F^s_\lambda$ implies $(t(a, C \cup b), A) \in F^s_\lambda$.

VIII. Let $p = t(a, B \cup C)$, $q = t(C, B)$ and $r = t(a \cup C, B)$, such that $(p, A \cup C) \in F_\lambda$ and $(q, A) \in F_\lambda$.

Claim 3.5. *Suppose p' and q' are such that $dom(p') = A \cup C$, $dom(q') = A$, are closed under conjuction, are $p' \vdash p$ and $q' \vdash q$. Then define r' by*

$$\{(\exists x_d)[\psi(x_a, x_c, x_d; b^1) \ \wedge \ \theta(x_c, x_d; b^2)] \mid b^1, b^2 \in A, c, d \in C, \psi(x_a; c, d, b^1) \in p', \theta(x_c, x_d; b^2) \in q'\}.$$

Then $r' \vdash r$.

Proof. Let b_0 and φ be such that $b_0 \in B$ and $\varphi(x_a, x_c, b_0) \in r$. There is $c \in c$ such that $a^\frown c \models \varphi(x_a, x_c, b_0)$. Therefore, $p' \vdash \varphi(x_a, c, b_0) \in p$. So, there is $\psi(x_a, c, d, b_1) \in p'$ with $d \in C$, $b_1 \in A \subseteq B$, and

$$\psi(x_a, c, d, b_1) \vdash \varphi(x_a, c, b_0).$$

Let $b_2 = b_1^\frown b_0$ and denote by $\varphi^*(x_c, x_d, b_2)$ the formula

$$\forall x_a[\psi(x_a, x_c, x_d, b_1) \to \varphi(x_a, x_c, b_0)].$$

Thus $c^\frown d \models \varphi^*(x_c, x_d, b_2)$, hence $\varphi^* \in q$. Since $q' \vdash q$, $q' \vdash \varphi^*$ and there are $b_3 \in A$ and $\theta(x_c, x_d, b_3) \in q'$ such that

$$\mathbb{M} \models \forall x_c \forall x_d[\theta(x_c, x_d, b_3) \to \varphi^*(x_c, x_d, b_2)].$$

We conclude that

$$\exists x_d[\psi(x_a, x_c, x_d, b_1) \wedge \theta(x_c, x_d, b_3)] \vdash \varphi(x_a, x_c, b_0),$$

the claim follows form the fact that $\exists x_d[\psi(x_a, x_c, x_d, b_1) \wedge \theta(x_c, x_d, b_3)] \in r'$. □

IX. Let us suppose, towards contradiction, that $p \in S(\bigcup_{i<\delta} B_i)$ and for all $i < \delta$, $(p \upharpoonright B_i, A) \in F_\lambda$, and $(p, A) \notin F_\lambda$. So there is $a \models p \upharpoonright A$ such that $a \not\models p$. Therefore, there is $\varphi(x) \in p$ such that a doesn't satisfies $\varphi(x)$. There is $i < \delta$, such that $\varphi(x) \in p \upharpoonright B_i$. Since $a \models p \upharpoonright A$ and $(p \upharpoonright B_i, A) \in F_\lambda$, $p \models\upharpoonright B_i$. Thus a satisfies $\varphi(x)$, a contradiction.

X. Let us suppose, towards contradiction, that there are p, A and B are such that $(p, A) \in F^s_\lambda$ and for all $q \in S(B)$ such that $p \subseteq q$, for all $A' \supseteq A$, $(q', A') \notin F^s_\lambda$. Therefore, for all $\eta \in 2^{\leq\lambda}$, we can find p_η, $A_\eta \subseteq B$ such that the following hold:

- $p_{()} = p \upharpoonright A$ and $A_{()} = A$;
- for all η, $p_\eta \in S(A_\eta)$, $A_{\eta^\frown(0)} = A_{\eta^\frown(1)}$ and $|A_{\eta^\frown(0)} - A_\eta| < \omega$;
- if η is an initial segment of ξ, then $p_\eta \subseteq p_\xi$;
- if $\alpha = length(\eta)$ is limit, then $p_\eta = \cup_{\beta<\alpha} p_{\eta\upharpoonright\beta}$;
- for all η, $p_{\eta^\frown(0)}$ is contradictory with $p_{\eta^\frown(1)}$.

Since $|A| < \lambda$ and $|A_{\eta^\frown(0)} - A_\eta| < \omega$ holds for all $\eta \in 2^{<\lambda}$, $B' = \cup_{\eta\in 2^{<\lambda}} A_\eta$ and $|B' \leq \lambda|$. It is clear that for all $\xi, \eta \in 2^\lambda$, p_ξ and p_η are contradictory. Thus $|S(B')| > \lambda$, a contradiction. □

Definition 3.6. *We say that a relation $R(x)$ of $\mathbb{M}$ is over A if it is definable by some formula $\varphi(x, a)$, $a \in A$.*

Definition 3.7. *We say that an equivalence relation $E(x, y)$ in $\mathbb{M}$ is finite, if the number of equivalence classes is finite.*

Let us denote by $FE(A)$ the set of all finite equivalence relations over A.

Definition 3.8. *We define $stp(a, A)$, the strong type of a over A, to be the set*

$$\{E(x, a) \mid E \in FE(A)\}.$$

Lemma 3.9 (Properties of strong types, [2], [12])**.** *Let $A \subseteq B$, a and b be arbitrary.*

1. *If $stp(a, A) = stp(b, A)$, $a \downarrow_A B$ and $b \downarrow_A B$, then $stp(a, B) = stp(b, B)$.*
2. *$stp(a, A) \vdash t(a, A)$.*
3. *If $stp(a, A) = stp(b, A)$, $a \downarrow_A B$ and $b \downarrow_A B$, then $t(a, B) = t(b, B)$.*
4. *If A is a model, then $t(a, A) \vdash stp(a, A)$.*
5. *If A is a model, $t(a, A) = t(b, A)$, $a \downarrow_A B$ and $b \downarrow_A B$, then $stp(a, B) = stp(b, B)$.*
6. *There is c such that $stp(c, A) = stp(a, A)$ and $c \downarrow_A B$.*

Definition 3.10. *Let us define F^a_λ to be the set of all pairs $(p, A) \in P_\lambda$ such that for some $a \models p$, $stp(a, A) \vdash p$.*

Lemma 3.11 ([2], [12])**.** *Let $(p, A) \in P_\lambda$. Then $(p, A) \in F^a_\lambda$ if and only if for all $a \models p$, $stp(a, A) \vdash p$.*

Proof. Let us suppose, towards contradiction, that there are $a, b \models p$ and c such that $stp(a, A) \vdash p$, $stp(b, A) = stp(c, A)$ and $c \not\models p$. Choose $f \in Aut(dom(p))$ such that $f(b) = a$. Let $a' = f(c)$. Then $stp(a', A) = stp(a, A)$ but $a' \not\models p$, a contradiction. □

Exercise 3.1. *Show that $F^t_\lambda \subseteq F^s_\lambda \subseteq F^a_\lambda$.*

Lemma 3.12 ([2], [12])**.** *If T is stable, then F^a_λ satisfies axioms I to IX. If T is a superstable countable theory over a countable vocabulary, then it also satisfies axiom X.*

Definition 3.13. *Let us define F^f_λ to be the set of all pairs $(p, A) \in P_\lambda$ that p does not fork over A.*

Lemma 3.14 ([2], [12])**.** *If T is stable, then F^f_λ satisfies axioms I to X.*

3.2 Models

Definition 3.15. *We say that $(A,(a_i,B_i)_{i<\alpha})$ is an F_λ-construction over A if for all $i<\alpha$, $(t(a_i,A_i),B_i)\in F_\lambda$, where $A_i=A\cup\bigcup_{j<i}a_j$. We say that C is F_λ-constructible over A if there is an F_λ-construction $(A,(a_i,B_i)_{i<\alpha})$ over A such that $C=A\cup\bigcup_{i<\alpha}a_i$.*

Definition 3.16. *We say that C is (F_λ,κ)-saturated if for all $B\subseteq C$ of size smaller than κ and $p\in S(B)$ the following holds:*

If A is such that $(p,A)\in F_\lambda$, then p is realized in C.

Definition 3.17. *We denote by $\mu(F_\lambda)$ the least cardinal μ such that for all $\kappa\geq\mu$ and C, if C is (F_λ,μ)-saturated, then it is (F_λ,κ)-saturated. If such μ does not exists, then we say $\mu(F_\lambda)=\infty$.*

Definition 3.18. *We say that (F_λ,κ)-primary over A if it is F_λ-constructible over A and (F_λ,κ)-saturated.*

Lemma 3.19 ([12])**.** *For all A and κ there is an (F_λ,κ)-primary set over A. If $\mu(F_\lambda)<\infty$, then there is an F_λ-primary set over A.*

Sketch of the proof. Let us construct by induction the sequence $\langle(a_\alpha,B_\alpha)\rangle$, such that $p_\alpha=t(a_\alpha,A_\alpha)$, $(p_\alpha,B_\alpha)\in F_\lambda$, where $A_\alpha=A\cup\bigcup\{a_i\mid i<\alpha\}$, and $A_\theta=A\cup\bigcup\{a_i\mid i<\theta\}$ is (F_λ,κ)-saturated.

Let α be such that for all $i<\alpha$ we have defined a_i and B_i. Suppose that A_α is not (F_λ,κ)-saturated. Therefore, there is at least one tuple $(q,C)\in F_\lambda$, $C\subseteq dom(q)\subseteq A_\alpha$, $|dom(q)|<\kappa$, such that q is not realized in A_α. Let us define for any (q,C) tuple of this kind,

$$j(q)=min(i|i\leq\alpha,dom(q)\subseteq A_i).$$

Let us choose (q_α,C_α) with $j(q_\alpha)$ minimal. By Axiom X, there is a pair $(p_\alpha,B_\alpha)\in F_\lambda$, such that $q_\alpha\subseteq p_\alpha$ and $p_\alpha\in S(A_\alpha)$. Finally choose a_α, such that $a_\alpha\models p_\alpha$. It is clear that if A_α is (F_λ,κ)-saturated we are done. Thus, it is enough to show that for some $\alpha<((|A|+2)^{\kappa+|T|})^+$, A_α is (F_λ,κ)-saturated. □

We say that C is F_λ-saturated if it is $(F_\lambda,|C|^+)$-saturated. Notice that C is F_λ-saturated if and only if C is $(F_\lambda,\mu(F_\lambda))$-saturated.

Definition 3.20. *We say that C is F_λ-primitive over A if for all F_λ-saturated $B\supseteq A$ there is an elementary embedding $f:C\to B$ such that $f\restriction A=id_A$.*

Definition 3.21. *We say that C is F_λ-prime over A if it is F_λ-primitive and F_λ-saturated.*

Lemma 3.22 ([12])**.** *If C is F_λ-constructible over A, then it is F_λ-primitive over A and so F_λ-primary sets over A are F_λ-prime over A.*

Lemma 3.23 ([12])**.** *If T is superstable, then for all λ and A, F^a_λ-prime models over A are F^a_λ-primary over A.*

Definition 3.24. *We say that C is F_λ-atomic over A if for all $c\in C$, there is $B\subseteq A$ such that $(t(c,A)),B)\in F_\lambda$.*

Lemma 3.25 ([2], [12])**.** *Let λ be a regular cardinal. If C is F_λ-constructible over A, then it is F_λ-atomic over A.*

Theorem 3.26 ([2], [12])**.** *Let λ be a regular cardinal. F_λ-primary sets over A are unique up to isomorphism over A (i.e. f an isomorphism with $f\restriction A=id_A$).*

4 Strong DOP

From now on we will work only with superstable theories.

4.1 Definitions

We will follow Section 2.2 of [9].

Definition 4.1. *We say that a model $\mathcal{M}$ is F_λ-minimal over A if $\mathcal{M}$ is F_λ-saturated and there is no F_λ-saturated model $\mathcal{N}$, $A\subseteq N\subsetneq M$.*

Definition 4.2. *A superstable theory T has the dimensional order property (DOP) if there are F^a_ω-saturated models $(M_i)_{i<3}$, $M_0\subset M_1\cap M_2$, $M_1\downarrow_{M_0}M_2$, and the F^a_ω-prime model over $M_1\cup M_2$ is not F^a_ω-minimal over $M_1\cup M_2$.*

Definition 4.3. *Let $p, q \in S(A)$. We say that p is orthogonal to q, $p \perp q$, if for all a, b and $B \supseteq A$ the following holds:*

If a realizes p, b realizes q, $a \downarrow_A B$, and $b \downarrow_A B$, then $a \downarrow_B b$.

Definition 4.4. *We say that $p \in S(A)$ is orthogonal to $B \subseteq A$, $p \perp B$, if p is orthogonal to every $q \in S(A)$ which does not fork over B.*

Fact 4.5 ([9], Fact 2.7)**.** *Let $B, D \subseteq M$, M a F^a_ω-saturated model over $B \cup D$, and $p \in S(M)$. If p is orthogonal to D and p does not fork over $B \cup D$, then for every $a \models p \restriction B \cup D$ the following holds: $a \downarrow_{B\cup D} M$ implies $tp(a, M) \perp D$.*

Proof. Notice that since M is a model, then every complete type over M is stationary. Let $p \in S(M)$ and $B, D \subseteq M$ such that p is orthogonal to D and p does not fork over $B \cup D$. Suppose, towards a contradiction, that there is a such that $a \models p \restriction B \cup D$, $a \downarrow_{B\cup D} M$ and $tp(a, M) \not\perp D$. Therefore, there are N and c, $D \subseteq N$, such that $a \downarrow_M N$, $c \downarrow_D M \cup N$, and $a \not\downarrow_N c$.
Let b be such that $b \models p$, there is $f \in Aut(\mathcal{M}, D \cup B)$ such that $f(a) = b$. Denote by N' the image $f(N)$. Choose b' such that $b' \downarrow_{B\cup D} M \cup N'$ and $stp(b', B \cup D) = stp(b, B \cup D)$. We know that $a \downarrow_{B\cup D} M$ and $a \downarrow_M N$, then by transitivity we get $a \downarrow_{B\cup D} M \cup N$. Therefore $a \downarrow_{B\cup D} N$, since $f \in Aut(\mathcal{M}, D \cup B)$ we conclude that $b \downarrow_{B\cup D} N'$. Since $stp(b', B \cup D) = stp(b, B \cup D)$ and $b' \downarrow_{B\cup D} N'$ we conclude that $tp(b, N' \cup B) = tp(b', N' \cup B)$, there is $h \in Aut(\mathcal{M}, N' \cup B)$ such that $h(b) = b'$. On the other hand, by the way we chose b, we know that $b \downarrow_{B\cup D} M$. Since $stp(b', B \cup D) = stp(b, B \cup D)$ and $b' \downarrow_{B\cup D} M$, then $tp(b', M) = tp(b, M) = p$. We conclude that there is $F \in Aut(\mathcal{M}, B \cup D)$ such that $F(a) = b'$ and $tp(b', M) \perp D$. Denote by c' the image $F(c)$.
Choose c'' such that $tp(c'', N' \cup B \cup b') = tp(c', N' \cup B \cup b')$ and $c'' \downarrow_{N'\cup B\cup b'} M$. Since $b' \downarrow_{B\cup N'} M$, then by transitivity we get $c''b' \downarrow_{N'\cup B} M$, so $c'' \downarrow_{N'\cup B} M$. On the other hand $c \downarrow_D M \cup N$, so $c \downarrow_D B \cup N$, since $F \in Aut(\mathcal{M}, B \cup D)$, we get $c' \downarrow_D B \cup N'$. By the way chose c'' we know that $tp(c'', N' \cup B) = tp(c', N' \cup B)$, therefore $c'' \downarrow_D B \cup N'$ and by transitivity we get $c'' \downarrow_D M \cup N'$.
We conclude that $c'' \downarrow_M N'$ and $c'' \downarrow_D M$, since $b' \downarrow_M N'$ and $tp(b', M) \perp D$, we get $b' \downarrow_{N'} c''$. By the way we chose c'' we know that $tp(c', N' \cup b') = tp(c'', N' \cup b')$, so $b' \downarrow_{N'} c'$. Since $F \in Aut(\mathcal{M}, B \cup D)$, we conclude that $a \downarrow_N c$, a contradiction. □

Fact 4.6. *[9] A type $p \in S(B \cup C)$ is orthogonal to C, if for every F^a_ω-primary model, M, over $B \cup C$ there exists a non-forking extension of p, $q \in S(M)$, orthogonal to C.*

Definition 4.7. *Let I be a set of infinite sequences. We say that I is indiscernible over A if for all $a_k, b_k \in I$, $k < n$, $a \in A$, and $\phi(x_0, \dots, x_{n-1}, y)$ the following holds:*

If for all $k < k', n$, $a_k \neq a_{k'}$, and $b_k \neq b_{k'}$, then

$$\mathbb{M} \models \phi(a_0, \dots, a_{n-1}, a) \leftrightarrow \phi(b_0, \dots, b_{n-1}, a).$$

Definition 4.8. *Let I be an infinite indiscernible set. We define $Av(I, A)$, the average type of I over A, to be the set:*

$$\{\varphi(x, a) \mid a \in A, |\{b \in I \mid \mathcal{M} \models \varphi(b, a)\}| \geq \omega\}.$$

Lemma 4.9 ([12], X.2 Lemma 2.2)**.** *Let $M_0 \subset M_1 \cap M_2$ be F^a_ω-saturated models, $M_1 \downarrow_{M_0} M_2$, M F^a_ω-atomic over $M_1 \cup M_2$ and F^a_ω-saturated. Then the following conditions are equivalent:*

1. *M is not F^a_ω-minimal over $M_1 \cup M_2$.*
2. *There is an infinite indiscernible $I \subseteq M$ over $M_1 \cup M_2$.*
3. *There is a type $p \in S(M)$ orthogonal to M_1 and to M_2, p not algebraic.*
4. *There is an infinite $I \subseteq M$ indiscernible over $M_1 \cup M_2$ such that $Av(I, M)$ is orthogonal to M_1 and to M_2.*

Lemma 4.10 ([6], Theorem 2.1)**.** *Let $M_0 \prec M_1, M_2$ be F^a_ω-saturated models, such that $M_1 \downarrow_{M_0} M_2$. Let M_3 be an F^a_ω-prime model over $M_1 \cup M_2$ and let $I \subseteq M_3$ be an indiscernible over $M_1 \cup M_2$ such that $Av(I, M_3)$ is orthogonal to M_1 and to M_2. If $(B_i)_{i<3}$ are sets such that:*

- *$B_0 \downarrow_{M_0} M_1 \cup M_2$.*
- *$B_1 \downarrow_{M_1\cup B_0} B_2 \cup M_2$.*
- *$B_2 \downarrow_{M_2\cup B_0} B_1 \cup M_1$.*

Then

$$tp(I, M_1 \cup M_2) \vdash tp(I, M_1 \cup M_2 \cup_{i<3} B_i).$$

Definition 4.11. *We say that $\{a_i \mid i<\alpha\}$ is independent over A if for all $i<\alpha$, $a_i \downarrow_A \cup\{a_j \mid j<\alpha, j\neq i\}$.*

Lemma 4.12 ([9], Lemma 2.10)**.** *Let $M_0 \subset M_1 \cap M_2$ be F^a_ω-saturated models, $M_1 \downarrow_{M_0} M_2$, M_3 F^a_ω-atomic over $M_1 \cup M_2$ and F^a_ω-saturated. Then the following conditions are equivalent:*

1. *There is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, that does not fork over $M_1 \cup M_2$.*
2. *There is an infinite indiscernible $I \subseteq M_3$ over $M_1 \cup M_2$ that is independent over $M_1 \cup M_2$.*
3. *There is an infinite $I \subseteq M_3$ indiscernible over $M_1 \cup M_2$ and independent over $M_1 \cup M_2$, such that $Av(I, M_3)$ is orthogonal to M_1 and to M_2.*

Proof. $3 \Rightarrow 2$. It is clear.

$3 \Rightarrow 1$. By 3, we know that there is an infinite $I \subseteq M_3$ indiscernible over $M_1 \cup M_2$ and independent over $M_1 \cup M_2$, such that $Av(I, M_3)$ is orthogonal to M_1 and to M_2. Then it is enough to show that for that I, indiscernible over $M_1 \cup M_2$, the type $Av(I, M_3)$ does not fork over $M_1 \cup M_2$.
Let b be such that $b \models Av(I, M_3)$, then $I \cup \{b\}$ is indiscernible over $M_1 \cup M_2$ and $b \downarrow_I M_3$.
Since I is independent over $M_1 \cup M_2$, then for all J finite subset of I and $a \in I\backslash J$, $a \downarrow_{M_1 \cup M_2} J$. Therefore $b \downarrow_{M_1\cup M_2} J$ holds for all J finite subset of I, because of the finite character we conclude that $b \downarrow_{M_1 \cup M_2} I$. By transitivity $b \downarrow_{M_1\cup M_2} M_3$, we conclude that $Av(I, M_3)$ does not fork over $M_1 \cup M_2$.

$2 \Rightarrow 3$. By 2, we know that there is an infinite indiscernible $I \subseteq M_3$ over $M_1 \cup M_2$ that is independent over $M_1 \cup M_2$. Then it is enough to show that for that I, indiscernible over $M_1 \cup M_2$, the type $Av(I, M_3)$ is orthogonal to M_1 and to M_2.
Suppose, towards a contradiction, that $Av(I, M_3)$ is not orthogonal to M_1. There is a countable set $J \subseteq I$, such that $Av(J, M_3)$ is not orthogonal to M_1. Therefore, $Av(J, J)$ is not orthogonal to some $r \in S(M_1)$.
Without loss of generality, we can assume that $J = J' \cup \{a_n | n<\omega\}$, for some $|J'|<\omega$ such that $Av(J, J)$ does not fork over J' and $Av(J, J')$ is stationary. Let $\{b_n | n<\omega\}$ be such that for every $n<\omega$, $b_n \models r$ and $b_n \downarrow_{M_1} M_1 \cup M_2 \cup J \cup \{b_m | m \neq n\}$. Therefore, there is $k<\omega$ such that $tp(a_0\frown\cdots\frown a_k, M_1 \cup M_2 \cup J')$ and $tp(b_0\frown\cdots\frown b_k, M_1\cup M_2 \cup J')$ are not weakly orthogonal.
Since M_3 is F^a_ω-atomic over $M_1 \cup M_2$, there is a finite $B \subseteq M_1 \cup M_2$ such that $stp(I', B) \vdash tp(I', M_1 \cup M_2)$, where $I' = J' \cup \{a_n | n<k\}$.
Since $tp(a_0\frown\cdots\frown a_k, M_1\cup M_2\cup J')$ is stationary and T is superstable, there is a finite set $C \subseteq M_1 \cup M_2$, that satisfies:

1. $tp(b_0\frown\cdots\frown b_k, C)$ is stationary.
2. $J' \cup B \cup (C\cap M_2) \cup b_0\frown\cdots\frown b_k \downarrow_{C\cap M_1} M_1$.
3. $a_0\frown\cdots\frown a_k \not\downarrow_{B\cup C\cup J'} b_0\frown\cdots\frown b_k$.

Since M_1 is F^a_ω-saturated, there is $c_0\frown\cdots\frown c_k \in M_1$ such that $stp(b_0\frown\cdots\frown b_k, C\cap M_1) = stp(c_0\frown\cdots\frown c_k, C\cap M_1)$. By the way $b_0\frown\cdots\frown b_k$ was chosen, we know that $b_n \downarrow_{M_1} M_1\cup M_2\cup J\cup\{b_m | m\neq n\}$ holds for every $n<\omega$, by transitivity we conclude that $b_0\frown\cdots\frown b_k \downarrow_{M_1} M_2 \cup J'$. By transitivity and the item 2 we get that $b_0\frown\cdots\frown b_k \downarrow_{C\cap M_1} M_1\cup M_2\cup J'$ and $b_0\frown\cdots\frown b_k \downarrow_{C\cap M_1} B\cup(C\cap M_2)\cup J'$. On the other hand, since $c_0\frown\cdots\frown c_k \in M_1$, by the item 2 we get that $c_0\frown\cdots\frown c_k \in M_1 \downarrow_{C\cap M_1} B\cup(C\cap M_2)\cup J'$. We conclude that $stp(b_0\frown\cdots\frown b_k, C\cup B\cup J') = stp(c_0\frown\cdots\frown c_k, C\cup B\cup J')$. Therefore, there is $f \in Saut(\mathcal{M}, C\cup B\cup J')$, such that $f(b_0\frown\cdots\frown b_k) = c_0\frown\cdots\frown c_k$.
By the item 3 we know that $a_0\frown\cdots\frown a_k \cup J' \not\downarrow_B b_0\frown\cdots\frown b_k \cup C$, so $f(a_0\frown\cdots\frown a_k)\cup J' \not\downarrow_B c_0\frown\cdots\frown c_k \cup C$ and $stp(f(a_0\frown\cdots\frown a_k)\cup J', B) \not\vdash tp(f(a_0\frown\cdots\frown a_k)\cup J', M_1\cup M_2)$. This contradicts that $stp(I', B)\vdash tp(I', M_1\cup M_2)$.
The same argument works to show that $Av(J, M_3)$ is orthogonal to M_2.

$1 \Rightarrow 3$. Let us show that $p \restriction B\cup M_1\cup M_2$ is realized in M_3, for every finite $B\subseteq M_3$, such that p does not fork over B and $p\restriction B$ is stationary.

Without loss of generality, we can assume that for $l \in 0,1,2$, $tp(B, M_l)$ does not fork over $B\cap M_l$ and $tp(B, M_1\cup M_2)$ does not fork over $B\cap(M_1\cup M_2)$. Let c realize p, $b_1\in M_1$, $b_2 \in M_2$, and $B_l = B\cap M_l$, for $l\in\{0,1,2\}$.

Since T is superstable, there is $C\subseteq M_0$ such that $b_l \downarrow_{B_l\cup C} M_0\cup B_l$, for $l\in 1,2$. Since $B\downarrow_{B_0} M_0$, then $B\downarrow_{B_0} C$ and $C\downarrow_{B_0} B$. Let q be a type over M_3 that extends $tp(C,B)$ and does not fork over B_0. Then $q\restriction M_1$ is orthogonal to p and parallel to $stp(C,B)$, so $stp(C,B)$ is orthogonal to p. Therefore

$$stp(c, B) \vdash stp(c, B\cup C). \tag{1}$$

Since $b_1 \cup B_1 \downarrow_{M_0} M_2$, then $b_1 \downarrow_{M_0 \cup B_1} M_2 \cup B_1$. By the way C was chosen, $b_1 \downarrow_{B_1 \cup C} M_0 \cup B_1$, by transitivity $b_1 \downarrow_{B_1 \cup C} M_2 \cup B_1$. Since $B_1 \cup C \subseteq M_2 \cup B$, then $stp(b_1, C \cup B)$ is parallel to some complete type over M_1 and orthogonal to p. Therefore $stp(b_1, C \cup B)$ is orthogonal to $stp(c, C \cup B)$ and

$$stp(c, C \cup B) \vdash stp(c, B \cup C \cup b_1). \tag{2}$$

Since $b_2 \cup B_2 \downarrow_{M_0} M_1$, then $b_2 \downarrow_{M_0 \cup B_1} M_1 \cup B_2$. By the way C was chosen, $b_2 \downarrow_{B_2 \cup C} M_0 \cup B_2$, by transitivity $b_1 \downarrow_{B_2 \cup C} M_1 \cup B_2$. Since $B_2 \cup C \subseteq M_1 \cup B$, then $stp(b_2, C \cup B \cup b_1)$ is parallel to some complete type over M_2 and orthogonal to p. Therefore $stp(b_2, C \cup B \cup b_1)$ is orthogonal to $stp(c, C \cup B \cup b_1)$ and

$$stp(c, C \cup B \cup b_1) \vdash stp(c, B \cup C \cup b_1 \cup b_2). \tag{3}$$

From (4), (5) and (6), we conclude $stp(c, B) \vdash stp(c, B \cup b_1 \cup b_2)$.
We conclude that $stp(c, B) \vdash p \restriction B \cup M_1 \cup M_2$. Since M_3 is F^a_ω-saturated and B is finite, then $p \restriction B \cup M_1 \cup M_2$ is realized in M_3.

Let $B_0 \subseteq M_1 \cup M_2$ be finite such that p does not fork over B_0 and $p \restriction B_0$ is stationary. We know that for every n there is $b_n \in M_3$ such that for $B_n = B_0 \cup \{b_i | i < n\}$, b_n realizes $p \restriction B_n \cup M_1 \cup M_2$. We conclude that $I = \{b_n | n < \omega\}$ is indiscernible and $p = Av(I, M_3)$, so $Av(I, M_3)$ is orthogonal to M_1 and to M_2.
To show that I is independent over $M_1 \cup M_2$, notice that for every $a \models p$, $a \downarrow_{M_1 \cup M_2} M_3$. Therefore, $a \downarrow_{M_1 \cup M_2} \{b_i | i < n\}$ holds for every $a \models p \restriction M_1 \cup M_2 \cup \{b_i | i < n\}$, especially $b_n \downarrow_{M_1 \cup M_2} \{b_i | i < n\}$. □

Definition 4.13. *We say that a superstable theory T has the strong dimensional order property (S-DOP) if the following holds:*
There are F^a_ω-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, such that $M_1 \downarrow_{M_0} M_2$, and for every M_3 F^a_ω-prime model over $M_1 \cup M_2$, there is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, such that it does not fork over $M_1 \cup M_2$.

In [5] Hrushovski and Sokolvić proved that the theory of differentially closed fields of characteristic zero (DCF) has eni-DOP, so it has DOP. The reader can find an outline of this proof in [8]. We will show that the models used in [8] also testify that the theory of differentially closed fields has S-DOP. We will focus on the proof of the S-DOP property:

There are F^a_ω-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, such that $M_1 \downarrow_{M_0} M_2$, and for every M_3 F^a_ω-prime model over $M_1 \cup M_2$, there is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, such that it does not fork over $M_1 \cup M_2$.

For more on DCF (proofs, definition, references) can be found in [7].

Definition 4.14. *A differential field is a field K with a derivation map $\delta : K \to K$ wit the properties:*

- $\delta(a + b) = \delta(a) + \delta(b)$
- $\delta(ab) = a\delta(b) + b\delta(a)$

We call $\delta(a)$ the derivative of a and we denote by $\delta^n(a)$ the nth derivative of a. For a differential field K we denote by $K\{x_1, x_2, \ldots, x_n\}$ the ring

$$K[x_1, x_2, \ldots, x_n, \delta(x_1), \delta(x_2), \ldots, \delta(x_n), \delta^2(x_1), \delta^2(x_2), \ldots, \delta^2(x_n), \ldots]$$

The derivation map δ is extended in $K\{x_1, x_2, \ldots, x_n\}$ by $\delta(\delta^m(x_i)) = \delta^{m+1}(x_i)$. We call $K\{x_1, x_2, \ldots, x_n\}$ the ring of differential polynomials over K.

Definition 4.15. *We say that a diferential field K is differentially closed if for any differential field $L \supseteq K$ and $f_1, f_2, \ldots, f_n \in K\{x_1, x_2, \ldots, x_n\}$ the system $f_1(x_1, x_2, \ldots, x_n) = f_2(x_1, x_2, \ldots, x_n) = f_n(x_1, x_2, \ldots, x_n) = 0$ has solution in L, then it has solution in K.*

Let K be a saturated model of DFC, $k \subseteq K$ and $a \in K^n$, we denote by $k\langle a\rangle$ the differentially closed subfield generated by $k(a)$. If $A \subseteq K$ and for all n, every nonzero $f \in k\{x_1, x_2, \ldots, x_n\}$, and all $a_1, a_2, \ldots, a_n \in A$ it holds that $f(a_1, a_2, \ldots, a_n) \neq 0$, then we say that A is δ-independent over k.
For all $k \subseteq K$ denote by k^{dif} the differential closure of k in K.

Theorem 4.16 (Hrushovski, Sokolvić)**.** *Suppose K_0 is a differentially closed field with characteristic zero, $\{a, b\}$ is δ-independent over K_0, $K_1 = K_0\langle a\rangle^{dif}$, $K_2 = K_0\langle b\rangle^{dif}$, $K = K_0\langle a, b\rangle^{dif}$, and p the non-forking extension of p_{a+b} in K. Then $K_1 \downarrow_{K_0} K_2$, $p \perp K_1$, and $p \perp K_2$.*

Corollary 4.17 ([9], Corollary 2.16)**.** *DFC has the S-DOP.*

Proof. Let a, b, K_1, K_2, and p be as in Theorem 4.16. By Theorem 4.16 it is enough to show that p does not fork over $K_1 \cup K_2$. By the way p was defined, we know that p does not fork over $a + b$, therefore p does not fork over $\{a, b\}$. Since $\{a, b\}$ is δ-independent over K_0, $K_1 = K_0\langle a\rangle^{dif}$, and $K_2 = K_0\langle b\rangle^{dif}$, we conclude that p does not fork over $K_1 \cup K_2$. □

4.2 Trees

Definition 4.18. *Let λ be an uncountable cardinal. A coloured tree is a pair (t, c), where t is a κ^+, $(\lambda+2)$-tree and c is a map $c : t_\lambda \to \kappa\backslash\{0\}$.*

Definition 4.19. *Let (t, c) be a coloured tree, suppose $(I_\alpha)_{\alpha<\kappa}$ is a collection of subsets of t that satisfies:*

- *for each $\alpha < \kappa$, I_α is a downward closed subset of t.*
- *$\bigcup_{\alpha<\kappa} I_\alpha = t$.*
- *if $\alpha < \beta < \kappa$, then $I_\alpha \subset I_\beta$.*
- *if γ is a limit ordinal, then $I_\gamma = \bigcup_{\alpha<\gamma} I_\alpha$.*
- *for each $\alpha < \kappa$ the cardinality of I_α is less than κ.*

We call $(I_\alpha)_{\alpha<\kappa}$ a filtration of t.

Order the set $\lambda\times\kappa\times\kappa\times\kappa\times\kappa$ lexicographically, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) > (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ if for some $1 \le k \le 5$, $\alpha_k > \beta_k$ and for every $i < k$, $\alpha_i = \beta_i$. Order the set $(\lambda \times \kappa \times \kappa \times \kappa \times \kappa)^{\le\lambda}$ as a tree by inclusion.
Define the tree (I_f, d_f) as, I_f the set of all strictly increasing functions from some $\theta \le \lambda$ to κ and for each η with domain λ, $d_f(\eta) = f(sup(rang(\eta)))$.
For every pair of ordinals α and β, $\alpha < \beta < \kappa$ and $i < \lambda$ define

$$R(\alpha, \beta, i) = \bigcup_{i<j\le\lambda} \{\eta : [i, j) \to [\alpha, \beta) | \eta \textit{ strictly increasing}\}.$$

Definition 4.20. *Assume κ is an inaccessible cardinal. If $\alpha < \beta < \kappa$ and $\alpha, \beta, \gamma \neq 0$, let $\{P^{\alpha,\beta}_\gamma | \gamma < \kappa\}$ be an enumeration of all downward closed subtrees of $R(\alpha, \beta, i)$ for all i, in such a way that each possible coloured tree appears cofinally often in the enumeration. And the tree $P^{0,0}_0$ is (I_f, d_f).*

This enumeration is possible because κ is inaccessible; there are at most $|\bigcup_{i<\lambda} \mathcal{P}(R(\alpha, \beta, i))| \le \lambda \times \kappa = \kappa$ downward closed coloured subtrees, and at most $\kappa \times \kappa^{<\kappa} = \kappa$ coloured trees.
Denote by $Q(P^{\alpha,\beta}_\gamma)$ the unique ordinal number i such that $P^{\alpha,\beta}_\gamma \subset R(\alpha, \beta, i)$.

Definition 4.21. *Assume κ is an inaccessible cardinal. Define for each $f \in \kappa^\kappa$ the coloured tree (J_f, c_f) by the following construction.*
For every $f \in \kappa^\kappa$ define $J_f = (J_f, c_f)$ as the tree of all $\eta : s \to \lambda \times \kappa^4$, where $s \le \lambda$, ordered by extension, and such that the following conditions hold for all $i, j < s$:
Denote by η_i, $1 \le i \le 5$, the functions from s to κ that satisfies, $\eta(n) = (\eta_1(n), \eta_2(n), \eta_3(n), \eta_4(n), \eta_5(n))$.

1. *$\eta \restriction n \in J_f$ for all $n < s$.*
2. *η is strictly increasing with respect to the lexicographical order on $\lambda \times \kappa^4$.*
3. *$\eta_1(i) \le \eta_1(i + 1) \le \eta_1(i) + 1$.*
4. *$\eta_1(i) = 0$ implies $\eta_2(i) = \eta_3(i) = \eta_4(i) = 0$.*
5. *$\eta_2(i) \ge \eta_3(i)$ implies $\eta_2(i) = 0$.*
6. *$\eta_1(i) < \eta_1(i + 1)$ implies $\eta_2(i + 1) \ge \eta_3(i) + \eta_4(i)$.*
7. *For every limit ordinal α, $\eta_k(\alpha) = sup_{\beta<\alpha}\{\eta_k(\beta)\}$ for $k \in \{1, 2\}$.*
8. *$\eta_1(i) = \eta_1(j)$ implies $\eta_k(i) = \eta_k(j)$ for $k \in \{2, 3, 4\}$.*
9. *If for some $k < \lambda$, $[i, j) = \eta_1^{-1}\{k\}$, then*

   $$\eta_5 \restriction [i, j) \in P^{\eta_2(i),\eta_3(i)}_{\eta_4(i)}.$$

   Note that 7 implies $Q(P^{\eta_2(i),\eta_3(i)}_{\eta_4(i)}) = i$.
10. *If $s = \lambda$, then either*

(a) *there exists an ordinal number m such that for every $k < m$ $\eta_1(k) < \eta_1(m)$, for every $k' \geq m$ $\eta_1(k) = \eta_1(m)$, and the color of η is determined by $P^{\eta_2(m),\eta_3(m)}_{\eta_4(m)}$:*

$$c_f(\eta) = c(\eta_5 \restriction [m, \lambda))$$

where c is the colouring function of $P^{\eta_2(m),\eta_3(m)}_{\eta_4(m)}$.

Or

(b) *there is no such ordinal m and then $c_f(\eta) = f(sup(rang(\eta_5)))$.*

The following lemma is a variation of Lemma 4.7 of [4]. In [4] Lemma 4.7 refers to trees of height $\omega + 2$ and the relation $=^\kappa_\omega$, nevertheless the proof is the same in both cases.

Lemma 4.22 ([9], Lemma 2.3)**.** *Suppose κ is an inaccessible cardinal. Then for every $f, g \in \kappa^\kappa$ the following holds*

$$f \ =^\kappa_\omega \ g \Leftrightarrow J_f \cong J_g$$

For each $\alpha < \kappa$ define J^α_f as

$$J^\alpha_f = \{\eta \in J_f | rang(\eta) \subset \lambda \times (\beta)^4 \text{ for some } \beta < \alpha\}.$$

Notice that $(J^\alpha_f)_{\alpha<\kappa}$ is a filtration of J_f and every $\eta \in J_f$ has the following properties:

1. $sup(rang(\eta_4)) \leq sup(rang(\eta_3)) = sup(rang(\eta_5)) = sup(rang(\eta_2))$.
2. When $\eta \restriction k \in J^\alpha_f$ holds for every $k \in \lambda$, $sup(rang(\eta_5)) \leq \alpha$. If in addition $\eta \notin J^\alpha_f$, then $sup(rang(\eta_5)) = \alpha$.

4.3 Constructing models

We will study only the superstable theories with S-DOP. Instead of write F^a_ω-constructible, F^a_ω-atomic, F^a_ω-saturated and F^a_ω-saturated we will write a-constructible, a-atomic, a-primary, a-prime and a-saturated. From now on T will be a superstable theory with S-DOP, unless otherwise stated. We will denote by λ the cardinal $(2^\omega)^+$.

Definition 4.23.

- *Let us define the dimension of a type $p \in S(A)$ in M by: $dim(p, M) = min\{|J| : J \subseteq M$, J is a maximal independent sequence over A, and $\forall a \in J, a \models p\}$*
- *Let us define the dimension of an indiscernible I over A in M by: $dim(I, A, M) = min\{|J| : J$ is equivalent to I and J is a maximal indiscernible over A in $M\}$. If for all J as above $dim(I, A, M) = |J|$, then we say that the dimension is true.*

Lemma 4.24 ([12], Lemma III 3.9)**.** *Let T be a superstable theory. If I is a maximal indiscernible set over A in M, then $|I| + \omega = dim(I, A, M) + \omega$, and if $dim(I, A, M) \geq \omega$, then the dimension is true.*

Theorem 4.25 ([12], Theorem IV 4.9)**.** *If M is an a-primary model over A, and $I \subseteq M$ is an infinite indiscernible set over A, then $dim(I, A, M) = \omega$.*

For any indiscernible sequence $I = \{a_i \mid i < \gamma\}$, we will denote by $I \restriction_\alpha$ the sequence $I = \{a_i \mid i < \alpha\}$. Since T has the S-DOP, there are a-saturated models $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of cardinality 2^ω and an indiscernible sequence $\mathcal{I}$ over $\mathcal{B} \cup \mathcal{C}$ of size κ that is independent over $\mathcal{B} \cup \mathcal{C}$ such that

1. $\mathcal{A} \subset \mathcal{B} \cap \mathcal{C}$, $\mathcal{B} \downarrow_\mathcal{A} \mathcal{C}$.
2. $Av(\mathcal{I}, \mathcal{B} \cup \mathcal{C})$ is orthogonal to $\mathcal{B}$ and to $\mathcal{C}$.
3. If $(B_i)_{i<3}$ are sets such that:
   (a) $B_0 \downarrow_\mathcal{A} \mathcal{B} \cup \mathcal{C}$.
   (b) $B_1 \downarrow_{\mathcal{B} \cup B_0} B_2 \cup \mathcal{C}$.
   (c) $B_2 \downarrow_{\mathcal{C} \cup B_0} B_1 \cup \mathcal{B}$.

   Then,
$$tp(\mathcal{I}, \mathcal{B} \cup \mathcal{C}) \vdash tp(\mathcal{I}, \mathcal{B} \cup \mathcal{C} \cup_{i<3} B_i).$$

By the existence property of forking, for any $D \supseteq \mathcal{A}$ there is $F \in Aut(\mathcal{A})$ such that for all $c \in \mathcal{B}$, $stp(F(c), \mathcal{A}) = stp(c, \mathcal{A})$ and $F(c) \downarrow_{\mathcal{A}} D$ (the same holds for $\mathcal{C}$). For every $\xi \in (J_f)_{<\lambda}$ and every $\eta \in (J_f)_\lambda$ ($(J_f)_\lambda$ are the elements of J_f at the level λ and $(J_f)_{<\lambda}$ are the elements of J_f at levels below λ), let $\mathcal{B}_\xi \cong_{\mathcal{A}} \mathcal{B}$, $\mathcal{A} \preceq \mathcal{B}_\xi$, and $\mathcal{C}_\eta \cong_{\mathcal{A}} \mathcal{C}$, $\mathcal{A} \preceq \mathcal{C}_\eta$, such that the models $(\mathcal{B}_\xi)_{\xi \in (J_f)_{<\lambda}}$ and $(\mathcal{C}_\eta)_{\eta \in (J_f)_\lambda}$ satisfy the following:

- $\mathcal{B}_\xi \downarrow_{\mathcal{A}} \bigcup\{\mathcal{B}_\zeta, \mathcal{C}_\theta \mid \zeta \in (J_f)_{<\lambda} \wedge \theta \in (J_f)_\lambda \wedge \zeta \neq \xi\}$.
- $\mathcal{C}_\eta \downarrow_{\mathcal{A}} \bigcup\{\mathcal{B}_\zeta, \mathcal{C}_\theta \mid \zeta \in (J_f)_{<\lambda} \wedge \theta \in (J_f)_\lambda \wedge \theta \neq \eta\}$.

We can choose this models due to the existence property and the finite character. Notice that all $\xi \in (J_f)_{<\lambda}$ and $\eta \in (J_f)_\lambda$, satisfy

$$\mathcal{B}_\xi \cup \mathcal{C}_\eta \downarrow_{\mathcal{A}} \bigcup\{\mathcal{B}_\zeta, \mathcal{C}_\theta \mid \zeta \in (J_f)_{<\lambda} \wedge \theta \in (J_f)_\lambda \wedge \zeta \neq \xi \wedge \theta \neq \eta\}.$$

Let $F_{\xi\eta}$ be an automorphism of the monster model such that $F_{\xi\eta} \restriction \mathcal{C} : \mathcal{C} \to \mathcal{C}_\eta$ and $F_{\xi\eta} \restriction \mathcal{B} : \mathcal{B} \to \mathcal{B}_\xi$ are isomorphisms and $F_{\xi\eta} \restriction \mathcal{A} = id$. Denote the sequence $\mathcal{I}$ by $\{w_\alpha \mid \alpha < \kappa\}$. For all $\eta \in (J_f)_\lambda$ and every $\xi < \eta$, let $I_{\xi\eta} = \{b_\alpha \mid \alpha < c_f(\eta)\}$ be an indiscernible sequence over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$ of size $c_f(\eta)$, that is independent over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$, and satisfies:

- $tp(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta) = tp(F_{\xi\eta}(\mathcal{I} \restriction c_f(\eta)), \mathcal{B}_\xi \cup \mathcal{C}_\eta)$.
- $I_{\xi\eta} \downarrow_{\mathcal{B}_\xi \cup \mathcal{C}_\eta} \bigcup\{\mathcal{B}_\zeta, \mathcal{C}_\theta \mid \zeta \in (J_f)_{<\lambda} \wedge \theta \in (J_f)_\lambda\} \cup \bigcup\{I_{\zeta\theta} \mid \zeta \neq \xi \vee \theta \neq \eta\}$.

To recap, $\mathcal{B}_\xi$, $\mathcal{C}_\eta$, and $I_{\xi\eta}$ satisfy the following:

1. $Av(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta)$ is orthogonal to $\mathcal{B}_\xi$ and to $\mathcal{C}_\eta$.
2. If $(B_i)_{i<3}$ are sets such that:
   (a) $B_0 \downarrow_{\mathcal{A}} \mathcal{B}_\xi \cup \mathcal{C}_\eta$.
   (b) $B_1 \downarrow_{\mathcal{B}_\xi \cup B_0} B_2 \cup \mathcal{C}_\eta$.
   (c) $B_2 \downarrow_{\mathcal{C}_\eta \cup B_0} B_1 \cup \mathcal{B}_\xi$.

   Then,
   $$tp(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta) \vdash tp(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup_{i<3} B_i).$$
3. $I_{\xi\eta} \downarrow_{\mathcal{B}_\xi \cup \mathcal{C}_\eta} \bigcup\{\mathcal{B}_\zeta, \mathcal{C}_\theta \mid \zeta \in (J_f)_{<\lambda} \wedge \theta \in (J_f)_\lambda\} \cup \bigcup\{I_{\zeta\theta} \mid \zeta \neq \xi \vee \theta \neq \eta\}$.

Definition 4.26. *Let Γ_f be the set $\bigcup\{\mathcal{B}_\xi, \mathcal{C}_\eta, I_{\xi\eta} \mid \xi \in (J_f)_{<\lambda} \wedge \eta \in (J_f)_\lambda \wedge \xi < \eta\}$ and let $\mathcal{A}^f$ be the a-primary model over Γ_f. Let Γ_f^α be the set $\bigcup\{\mathcal{B}_\xi, \mathcal{C}_\eta, I_{\xi\eta} \mid \xi, \eta \in J_f^\alpha \wedge \xi < \eta\}$.*

Fact 4.27 ([9], Fact 3.6)**.** *If α is such that $\alpha^\lambda < f(\alpha)$, $sup(\{c_f(\eta)\}_{\eta \in J_f^\alpha}) < \alpha$, then $|\Gamma_f^{\alpha+1}| = f(\alpha)$.*

Lemma 4.28 ([9], Lemma 3.7)**.** *For every $\xi \in (J_f)_{<\lambda}$, $\eta \in (J_f)\lambda$, $\xi < \eta$, let $p_{\xi\eta}$ be the type $Av(I_{\xi\eta} \restriction \omega, I_{\xi\eta} \restriction \omega \cup \mathcal{B}_\xi \cup \mathcal{C}_\eta)$. If $c_f(\eta) > \omega$, then $dim(p_{\xi\eta}, \mathcal{A}^f) = c_f(\eta)$.*

Proof. Denote by S the set $I_{\xi\eta} \restriction \omega \cup \mathcal{B}_\xi \cup \mathcal{C}_\eta$, so $p_{\xi\eta} = Av(I_{\xi\eta} \restriction \omega, S)$.
Suppose, towards a contradiction, that $dim(p_{\xi\eta}, \mathcal{A}^f) \neq c_f(\eta)$. Since $I_{\xi\eta} \subset \mathcal{A}^f$, then $dim(p_{\xi\eta}, \mathcal{A}^f) > c_f(\eta)$. Therefore, there is an independent sequence $I = \{a_i | i < c_f(\eta)^+\}$ over S such that $I \subset \mathcal{A}^f$ and $\forall a \in I$, $a \models p_{\xi\eta}$.

Claim 4.29. *$I_{\xi\eta} \restriction \omega \cup I$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$.*

Proof. We will show by induction on α, that $I_{\xi\eta} \restriction \omega \cup \{a_i | i \leq \alpha\}$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$.
Case $\alpha = 0$.
Since $a_0 \models p_{\xi\eta}$, then $tp(a_0, S) = Av(I_{\xi,\eta} \restriction \omega, S)$ and $I_{\xi\eta} \restriction \omega \cup \{a_0\}$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$.

Suppose α is an ordinal such that for every $\beta < \alpha$, $I_{\xi\eta} \restriction \omega \cup \{a_i | i \leq \beta\}$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$. Therefore, $I_{\xi\eta} \restriction \omega \cup \{a_i | i < \alpha\}$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$. By the way I was chosen, we know that $a_\alpha \downarrow_S \{a_i | i < \alpha\}$ and $a_\alpha \models p_{\xi\eta}$. Since $I_{\xi\eta} \restriction \omega \cup \{a_i | i < \alpha\}$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$, then $Av(I_{\xi\eta} \restriction \omega, S \cup \{a_i | i < \alpha\}) = Av(I_{\xi\eta} \restriction \omega \cup \{a_i | i < \alpha\}, S \cup \{a_i | i < \alpha\})$, therefore $Av(I_{\xi\eta} \restriction \omega \cup \{a_i | i < \alpha\}, S \cup \{a_i | i < \alpha\})$ does not fork over S. Since $Av(I_{\xi\eta} \restriction \omega \cup \{a_i | i < \alpha\}, S \cup \{a_i | i < \alpha\})$ is stationary, we conclude that $tp(a_\alpha, S \cup \{a_i | i < \alpha\}) = Av(I_{\xi,\eta} \restriction \omega \cup \{a_i | i < \alpha\}, S \cup \{a_i | i < \alpha\})$ and $I_{\xi,\eta} \restriction \omega \cup \{a_i | i \leq \alpha\}$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$. $\square$

In particular $I_{\xi\eta} \restriction \omega \cup I$ is indiscernible, and $I_{\xi\eta}$ is equivalent to I.

Claim 4.30. *$tp(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta) \vdash tp(I_{\xi\eta}, \Gamma_f \backslash I_{\xi\eta})$ and $I_{\xi\eta}$ is indiscernible over $\Gamma_f \backslash I_{\xi\eta}$.*

Proof. Define:

$$B_0 = \bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\} \cup \bigcup\{I_{rp} | r \neq \xi \wedge p \neq \eta\}$$
$$B_1 = \bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\} \cup \bigcup\{I_{rp} | p \neq \eta\}$$
$$B_2 = \bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\} \cup \bigcup\{I_{rp} | r \neq \xi\}$$

Notice that by the way we chose the sequences I_{xy}, for every $r < p$ it holds that

$$I_{rp} \downarrow_{\mathcal{B}_r \cup \mathcal{C}_p} \bigcup\{\mathcal{B}_\zeta, \mathcal{C}_\theta | \zeta, \theta \in J_f\} \cup \bigcup\{I_{\zeta\theta} | \zeta \neq r \vee \theta \neq p\}.$$

Let J be a finite subset of $\{I_{rp} | r \neq \xi \wedge p \neq \eta\}$, $J = \{I_i | i < m\}$, then

$$I_0 \downarrow_{\bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\}} \mathcal{B}_\xi \cup \mathcal{C}_\eta$$

and

$$I_1 \downarrow_{\bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\} \cup I_0} \mathcal{B}_\xi \cup \mathcal{C}_\eta,$$

by transitivity

$$I_0 \cup I_1 \downarrow_{\bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\}} \mathcal{B}_\xi \cup \mathcal{C}_\eta.$$

In general, if $n < m - 1$ is such that

$$\{I_i | i \leq n\} \downarrow_{\bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\}} \mathcal{B}_\xi \cup \mathcal{C}_\eta,$$

then since

$$I_{n+1} \downarrow_{\bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\} \cup \bigcup\{I_i | i \leq n\}} \mathcal{B}_\xi \cup \mathcal{C}_\eta$$

we conclude by transitivity that

$$\{I_i | i \leq n+1\} \downarrow_{\bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\}} \mathcal{B}_\xi \cup \mathcal{C}_\eta.$$

We conclude

$$\bigcup J \downarrow_{\bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\}} \mathcal{B}_\xi \cup \mathcal{C}_\eta.$$

Because of the finite character we get that

$$\bigcup\{I_{rp} | r \neq \xi \wedge p \neq \eta\} \downarrow_{\bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\}} \mathcal{B}_\xi \cup \mathcal{C}_\eta.$$

By the way we chose the models $\mathcal{B}_x$ and $\mathcal{C}_y$, we know that

$$\mathcal{B}_\xi \cup \mathcal{C}_\eta \downarrow_{\mathcal{A}} \bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\},$$

by transitivity we conclude $B_0 \downarrow_{\mathcal{A}} \mathcal{B}_\xi \cup \mathcal{C}_\eta$.
Notice that for every $p \neq \eta$, $\xi < p$ we have

$$I_{\xi p} \downarrow_{\mathcal{B}_\xi \cup \mathcal{C}_p} \bigcup\{\mathcal{B}_\zeta, \mathcal{C}_\theta | \zeta, \theta \in J_f\} \cup \bigcup\{I_{\zeta\theta} | \zeta \neq \xi \vee \theta \neq p\}$$

so

$$I_{\xi p} \downarrow_{\mathcal{B}_\xi \cup B_0} \mathcal{C}_\eta \cup \bigcup\{I_{\zeta\theta} | \zeta \neq \xi \vee \theta \neq p\}.$$

From this we can conclude, in a similar way as before, that for every finite $J \subseteq \{I_{\xi p} | p \neq \eta\}$ it holds that

$$\bigcup J \downarrow_{\mathcal{B}_\xi \cup B_0} \mathcal{C}_\eta \cup \bigcup\{I_{\zeta\theta} | \zeta \neq \xi\}.$$

Because of the finite character we get that

$$\bigcup\{I_{\xi p} | p \neq \eta\} \downarrow_{\mathcal{B}_\xi \cup B_0} \mathcal{C}_\eta \cup \bigcup\{I_{\zeta\theta} | \zeta \neq \xi\}.$$

Since $\bigcup\{\mathcal{B}_r \cup \mathcal{C}_p | r \neq \xi \wedge p \neq \eta\} \subseteq B_0$ and $\bigcup\{I_{rp} | r \neq \xi \wedge p \neq \eta\} \subseteq B_0$, then we conclude

$$B_1 \downarrow_{\mathcal{B}_\xi \cup B_0} \mathcal{C}_\eta \cup B_2.$$

Using a similar argument, it can be proved that

$$B_2 \downarrow_{\mathcal{C}_\eta \cup B_0} \mathcal{B}_\xi \cup B_1.$$

To summary, the following holds:

- $B_0 \downarrow_{\mathcal{A}} \mathcal{B}_\xi \cup \mathcal{C}_\eta$,
- $B_1 \downarrow_{\mathcal{B}_\xi \cup B_0} \mathcal{C}_\eta \cup B_2$,
- $B_2 \downarrow_{\mathcal{C}_\eta \cup B_0} \mathcal{B}_\xi \cup B_1$,

by the way the sequences I_{xy} were chosen (item 2), we can conclude that $tp(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta) \vdash tp(I_{\xi\eta}, \Gamma_f \backslash I_{\xi\eta})$ and since $I_{\xi\eta}$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$, then $I_{\xi\eta}$ is indiscernible over $\Gamma_f \backslash I_{\xi\eta}$. □

By Claim 4.7.1 we know that $tp(I, \mathcal{B}_\xi \cup \mathcal{C}_\eta) = tp(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta)$, therefore by Claim 4.7.2 $tp(I, \mathcal{B}_\xi \cup \mathcal{C}_\eta) \vdash tp(I_{\xi\eta}, \Gamma_f \backslash I_{\xi\eta})$. We conclude that $tp(I, \mathcal{B}_\xi \cup \mathcal{C}_\eta) \vdash tp(I, \Gamma_f \backslash I_{\xi\eta})$ and since I is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$, then I is indiscernible over $\Gamma_f \backslash I_{\xi\eta}$.

Claim 4.31. *There are $I', I^* \subseteq I$ such that $|I'| = c_f(\eta)^+$ and $I' \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^*} I_{\xi\eta}$.*

Proof. Let us denote the elements of $I_{\xi\eta}$ by b_i, $I_{\xi\eta} = \{b_i | i < c_f(\eta)\}$. Since T is superstable, we know that for every $\alpha < c_f(\eta)$ there is a finite $B_\alpha \subseteq I \cup \{b_i | i < \alpha\}$ such that $b_\alpha \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup B_\alpha} I \cup \{b_i | i < \alpha\}$. Define $I^* = (\bigcup_{\alpha < c_f(\eta)} B_\alpha) \cap I$ and $I' = I \backslash I^*$, notice that $|I^*| \leq c_f(\eta)$, so $|I'| = c_f(\eta)^+$. Because of the finite character, to prove that $I' \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^*} I_{\xi\eta}$, it is enough to prove that $I' \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^*} \{b_i | i < \alpha\}$ holds for every $\alpha < c_f(\eta)$. Let us prove this by induction on $\alpha > 0$.

Case: $\alpha = 1$.
By the way B_0 was chosen, we know that $b_0 \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup B_0} I$, and this implies $I' \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^*} b_0$.

Case: $\alpha = \beta + 1$.
Suppose β is such that $I' \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^*} \{b_i | i < \beta\}$ holds. By the way B_β was chosen, we know that $b_\beta \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup B_\beta} I \cup \{b_i | i < \beta\}$ and $B_\beta \subseteq I \cup \{b_i | i < \beta\}$. Therefore $b_\beta \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^* \cup \{b_i | i < \beta\}} I'$ and by the induction hypothesis and transitivity, we conclude that $\{b_i | i \leq \beta\} \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^*} I'$. So $I' \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^*} \{b_i | i < \alpha\}$.

Case: α is a limit ordinal.
Suppose α is a limit ordinal such that $I' \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^*} \{b_i | i < \beta\}$ holds for every $\beta < \alpha$. Therefore, for every finite $A \subseteq \{b_i | i < \alpha\}$ we know that $I' \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^*} A$. Because of the finite character, we conclude that $I' \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^*} \{b_i | i < \alpha\}$. □

Claim 4.32. *I' is is indiscernible over $\Gamma_f \cup I^*$, in particular I' is is indiscernible over Γ_f.*

Proof. Let $\{c_0, c_1, \dots, c_n\}$ and $\{c'_0, c'_1, \dots, c'_n\}$ be disjoint subsets of I' with n elements, such that $i \neq j$ implies $c_i \neq c_j$ and $c'_i \neq c'_j$. We will prove that the following holds for every $m \leq n$

$$tp(\{c'_0, \dots, c'_{m-1}, c_m, c_{m+1}, c_n\}, \Gamma_f \cup I^*) = tp(\{c'_0, \dots, c'_{m-1}, c'_m, c_{m+1}, \dots, c_n\}, \Gamma_f \cup I^*).$$

By Claim 4.7.3, we know that $\{c_0, c_1, \dots, c_n\} \cup \{c'_0, c'_1, \dots, c'_n\} \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^*} I_{\xi\eta}$, so $c_m \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^* \cup \{c'_0, \dots, c'_{m-1}, c_{m+1}, \dots, c_n\}} I_{\xi\eta}$ and $c'_m \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^* \cup \{c'_0, \dots, c'_{m-1}, c_{m+1}, \dots, c_n\}} I_{\xi\eta}$.
Since $\{c_m, c'_m\} \cup I^* \cup \{c'_0, \dots, c'_{m-1}, c_{m+1}, \dots, c_n\}$ is indiscernible over $(\Gamma_f \backslash I_{\xi\eta})$, and $\{c_0, c_1, \dots, c_n\} \cap \{c'_0, c'_1, \dots, c'_n\} = \emptyset$, then

$$c_m \models Av(I^* \cup \{c'_0, \dots, c'_{m-1}, c_{m+1}, \dots, c_n\}, (\Gamma_f \backslash I_{\xi\eta}) \cup I^* \cup \{c'_0, \dots, c'_{m-1}, c_{m+1}, \dots, c_n\})$$

and

$$c'_m \models Av(I^* \cup \{c'_0, \dots, c'_{m-1}, c_{m+1}, \dots, c_n\}, (\Gamma_f \backslash I_{\xi\eta}) \cup I^* \cup \{c'_0, \dots, c'_{m-1}, c_{m+1}, \dots, c_n\}).$$

We know that $Av(I^* \cup \{c'_0, \dots, c'_{m-1}, c_{m+1}, \dots, c_n\}, (\Gamma_f \backslash I_{\xi\eta}) \cup I^* \cup \{c'_0, \dots, c'_{m-1}, c_{m+1}, \dots, c_n\})$ is stationary, we conclude that

$$tp(c_m, \Gamma_f \cup I^* \cup \{c'_0, \dots, c'_{m-1}, c_{m+1}, \dots, c_n\}) = tp(c'_m, \Gamma_f \cup I^* \cup \{c'_0, \dots, c'_{m-1}, c_{m+1}, \dots, c_n\})$$

and

$$tp(\{c'_0, \dots, c'_{m-1}, c_m, c_{m+1}, \dots, c_n\}, \Gamma_f \cup I^*) = tp(\{c'_0, \dots, c'_{m-1}, c'_m, c_{m+1}, \dots, c_n\}, \Gamma_f \cup I^*)$$

as we wanted.
Since

$$tp(\{c'_0, \dots, c'_{m-1}, c_m, c_{m+1}, \dots, c_n\}, \Gamma_f \cup I^*) = tp(\{c'_0, \dots, c'_{m-1}, c'_m, c_{m+1}, \dots, c_n\}, \Gamma_f \cup I^*)$$

holds for every $m \leq n$, we conclude that

$$tp(\{c_0, \dots, c_n\}, \Gamma_f \cup I^*) = tp(\{c'_0, \dots, c'_n\}, \Gamma_f \cup I^*).$$

To finish the proof, let $\{c_0, c_1, \dots, c_n\}$ and $\{c'_0, c'_1, \dots, c'_n\}$ be subsets of I' with n elements, such that $i \neq j$ implies $c_i \neq c_j$ and $c'_i \neq c'_j$. Since I' is infinite, then there is $\{c''_0, c''_1, \dots, c''_n\} \subseteq I'$ such that $\{c''_0, c''_1, \dots, c''_n\} \cap (\{c_0, c_1, \dots, c_n\} \cup \{c'_0, c'_1, \dots, c'_n\}) = \emptyset$. Therefore

$$tp(\{c_0, \dots, c_n\}, \Gamma_f \cup I^*) = tp(\{c''_0, \dots, c''_n\}, \Gamma_f \cup I^*) = tp(\{c'_0, \dots, c'_n\}, \Gamma_f \cup I^*),$$

we conclude that I' is is indiscernible over $\Gamma_f \cup I^*$. □

Let $J \subset \mathcal{A}^f$ be a maximal indiscernible set over Γ_f such that $I' \subseteq J$. By Lemma 4.2 $|J| + \kappa(T) = dim(J, \Gamma_f, \mathcal{A}^f) + \kappa(T)$. Since T is superstable, $\kappa(T) < \omega < |J|$ and we conclude that $\kappa(T) < dim(J, \Gamma_f, \mathcal{A}^f) + \kappa(T)$. Therefore $\kappa(T) < dim(J, \Gamma_f, \mathcal{A}^f)$ and by Lemma 4.2 the dimension is true, $dim(J, \Gamma_f, \mathcal{A}^f) = |J|$. So $dim(J, \Gamma_f, \mathcal{A}^f) > \omega$ a contradiction with Theorem 4.3. □

Theorem 4.33 ([9], Theorem 4.1)**.** *Assume f, g are functions from κ to $Card \cap \kappa \backslash \lambda$ such that $f(\alpha), g(\alpha) > \alpha^{++}$ and $f(\alpha), g(\alpha) > \alpha^{\lambda}$. Then $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic if and only if f and g are $=^\kappa_\lambda$ equivalent.*

This lemma has a long proof we will sketch the proof. One direction is easy, for the other direction we proceed by contradiction, we assume that f and g are not $=^\kappa_\lambda$ equivalent and there is an isomorphism $\Pi : \mathcal{A}^f \to \mathcal{A}^g$. Then we construct an a-primary model F, and find $\xi < \eta$ and $a \in I_{\xi\eta}$ such that

$$\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)} F.$$

By using a, we will construct a independent indiscernible sequence $(b_i)_{i<f(\alpha)^+}$ over $\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)$ in $\mathcal{A}^g$. Finally, we use Π and $(b_i)_{i<f(\alpha)^+}$ to construct a sequence $(c_i)_{i<f(\alpha)^+}$ indiscernible and independent over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$ with $c_0 \in I_{\xi\eta}$, which is a contradiction with Lemma 4.28.

Proof. **Sketch.** From right to left. If f and g are $=^\kappa_\lambda$ equivalent then J_f and J_g are isomorphic. Let $G : J_f \to J_g$ a colored trees isomorphism, the proof follows by showing that G defines an embedding $H : \Gamma_f \to \Gamma_g$ and this one can be extended to an isomorphism between $\mathcal{A}^f$ and $\mathcal{A}^g$.

From left to right. For every α define $\mathcal{A}^\alpha_f = \Gamma^\alpha_f \cup \bigcup\{a^f_i \mid i < \alpha\}$, clearly $\mathcal{A}^\alpha_f$ is not necessary a model. Suppose that $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic but f and g are not $=^\kappa_\lambda$ equivalent.

Denote by $\Pi : \mathcal{A}^f \to \mathcal{A}^g$ an isomorphism. There are α and η such that $f(\alpha) > g(\alpha)$, $\Pi(\mathcal{A}^\alpha_f) = \mathcal{A}^\alpha_g$ and $c_f(\eta) = f(\alpha)$. There is $X \subset \Gamma_g$ of size 2^ω such that $\Pi(\mathcal{C}_\eta) \subset D$, where D is the a-primary model over X. There is $\beta < \alpha$ such that $X \cap \Gamma^\alpha_f \subset \Gamma^\beta_f$, and ξ such that $\mathcal{B}_\xi \subset \Gamma^\alpha_f \backslash \Gamma^\beta_f$.

Denote by E the a-primary model over $X \cup \Gamma^{\alpha+1}_g$. There is $a \in I_{\xi\eta}$ such that $\Pi(a) \notin E$ and $\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)} F$, where F is the a-primary model over $E \cup \bigcup\{\mathcal{B}_\zeta, I_{\zeta\theta} \mid \zeta < \theta \wedge \mathcal{C}_\theta \subseteq X \backslash \Gamma^{\alpha+1}_g\}$.

Since $\mathcal{A}_g$ is a-atomic, there is a finite $B \subseteq F \cup \Gamma_g$ such that $(tp(\Pi(a), F \cup \Gamma_g), B) \in F^a_\omega$. There is $\mathcal{Y}$ such that $B \backslash F \subset \mathcal{Y}$ and $S = \{r \in J_g \mid (r \in (J_g)_{<\lambda} \wedge \mathcal{B}_r \subset \mathcal{Y}) \vee (r \in (J_g)_\lambda \wedge \mathcal{C}_r \subset \mathcal{Y})\}$ is finite.

Let $\bar{S}$ be the smallest subtree of J_g that is closed under predecessors and contains S. Define $\mathcal{X} = \{r \in J_g \mid (r \in (J_g)_{<\lambda} \wedge \mathcal{B}_r \subset X) \vee (r \in (J_g)_\lambda \wedge \mathcal{C}_r \subset X)\}$ and $\bar{\mathcal{X}}$ as the smallest subtree of J_g that is closed under predecessors and contains $\mathcal{X}$. Let $\{u_i\}_{i<f(\alpha)^+}$ be a sequence of subtrees of J_g with the following properties:

- $u_0 = \bar{S}$
- Every u_i is a tree isomorphic to u_0.
- If $i \neq j$, then $u_i \cap u_j = u_0 \cap (\bar{\mathcal{X}} \cup J^{\alpha+1}_g)$.
- Every $\zeta \in dom(c_g) \cap u_0$ satisfies $c_g(\zeta) = c_g(G_i(\zeta))$, where G_i is the isomorphism between u_0 and u_i.

With these subtrees we can find a sequence $\{b_i\}_{i<f(\alpha)^+}$ of elements of $\mathcal{A}^g$ such that for all $i < f(\alpha)^+$, $tp(b_i, F) = tp(\Pi(a), F)$ and $b_i \downarrow_F \bigcup_{j<i} b_j$. Since $\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)} F$, then $b_i \downarrow_{\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)} \bigcup_{j<i} b_j$ holds for all $i < f(\alpha)^+$.

We conclude that $(b_i)_{i<f(\alpha)^+}$ is an indiscernible sequence over $\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)$ and independent over $\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)$. Since Π is an isomorphism, we obtain in $\mathcal{A}^f$ a sequence $(c_i)_{i<f(\alpha)^+}$ indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$ and independent over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$. So $dim(p_{\xi\eta}, \mathcal{A}^f) \geq f(\alpha)^+$ a contradiction with Lemma 4.28 ($dim(p_{\xi\eta}, \mathcal{A}^f) = f(\alpha)$).

□

Lemma 4.34 ([9], Corollary 5.1)**.** *If κ is inaccessible, and T is a theory with S-DOP, then $=^\kappa_\lambda \hookrightarrow_c \cong_T$.*

Theorem 4.35 ([9], Corollary 5.2)**.** *If κ is an inaccessible and T_1 is a classifiable theory and T_2 is a superstable theory with S-DOP, then $\cong_{T_1} \hookrightarrow_c \cong_{T_2}$.*

5 Questions

Question 5.1. *Let κ be an inaccessible cardinal, T_1 a classifiable theory, and T_2 a non-classifiable theory. Is $\cong_{T_1} \hookrightarrow_c \cong_{T_2}$ a theorem of ZFC?*

Question 5.2 (J. Baldwin)**.** *Does there exists a superstable theory with DOP that does not have S-DOP?*

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