Topics in Logic: Generalized Descriptive Set Theory

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Contents

1 Generalized Baire spaces

Generalized descriptive set theory is the generalization of descriptive set theory to uncountable cardinals. For a background on classical descriptive set theory see [11] or [12]. We will denote by κ^κ the set of functions $f:\kappa\to\kappa$, 2^κ the set of functions $f:\kappa\to 2$, and $\kappa^{<\kappa}$ the set of functions $f:\alpha\to\kappa$ where $\alpha<\kappa$. During these notes, κ will be an uncountable cardinal that satisfies $\kappa^{<\kappa}=\kappa$, unless otherwise is stated.

The aim of this first section is to introduce the notions of κ-Borel class, $\Delta_1^1(\kappa)$ class, κ-Borel* class, and show the relation between these classes.

1.1 Topology

Definition 1.1. *$\mathcal{I}\subseteq\mathcal{P}(\kappa)$ is an ideal if the following holds:*

- $\mathcal{I}\neq\emptyset$,
- *for all $x\in\mathcal{I}$, if $y\subseteq x$, then $y\in\mathcal{I}$,*
- *if $x,y\in\mathcal{I}$, then $x\cup y\in\mathcal{I}$.*

An ideal $\mathcal{I}$ is $<\kappa$-complete if it is closed under the union of size less than κ. An ideal $\mathcal{I}$ is proper if $\mathcal{I}\neq\mathcal{P}(\kappa)$.

Example 1.1. *The set of bounded subsets of κ, $\{X\subseteq\kappa\mid\exists\alpha<\kappa\forall\beta\in X(\beta<\alpha)\}$, form an ideal.*

Definition 1.2 (Ideal topology)**.** *Let $\mathcal{I}$ be a $<\kappa$-complete proper ideal on κ that extends the ideal of bounded sets. The ideal topology associated to $\mathcal{I}$ is the topology generated by the following basic open sets. For every $A\in\mathcal{I}$, $\xi\in\kappa^A$ we define the basic open set N_ξ by*

$$N_\xi=\{\eta\in\kappa^\kappa\mid\xi\subseteq\eta\}.$$

The open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

Definition 1.3 (The Generalized Baire space $\mathbf{B}(\kappa)$)**.** *Let κ be an uncountable cardinal. The generalized Baire space is the set κ^κ endowed with the following topology. For every $\eta\in\kappa^{<\kappa}$, define the following basic open set*

$$N_\eta=\{f\in\kappa^\kappa\mid\eta\subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

Definition 1.4 (The Generalized Cantor space $\mathbf{C}(\kappa)$)**.** *Let κ be an uncountable cardinal. The generalized Cantor space is the set 2^κ endowed with the following topology. For every $\eta\in 2^{<\kappa}$, define the following basic open set*

$$N_\eta=\{f\in 2^\kappa\mid\eta\subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

Exercise 1.1. *Show that the topology in the previous definition is the ideal topology associated to the ideal of bounded sets.*

1.2 Borel sets

Definition 1.5 (κ-Borel class)**.** *Let $S\in\{\mathbf{B}(\kappa),\mathbf{C}(\kappa)\}$. The class κ-Borel(S) of all κ-Borel sets in S is the least collection of subsets of S which contains all open sets and is closed under complements, unions and intersections both of length at most κ.*

Definition 1.6. *Let us define the following hierarchy.*

- $\Sigma_1^0=\{X\subseteq\kappa^\kappa\mid X\text{ is open}\}$
- $\Pi_1^0=\{X\subseteq\kappa^\kappa\mid X\text{ is closed}\}$
- $\Sigma_\alpha^0=\{\bigcup_{\gamma<\kappa}A_\gamma\mid A_\gamma\in\bigcup_{1\leq\beta<\alpha}\Pi_\beta^0\}$
- $\Pi_\alpha^0=\{\kappa^\kappa\backslash X\mid X\in\Sigma_\alpha^0\}$

Exercise 1.2. *Show that κ-Borel$=\bigcup_{\alpha<\kappa^+}\Sigma_\alpha^0$.*

Exercise 1.3. *Let $S\in\{\mathbf{B}(\kappa),\mathbf{C}(\kappa)\}$ and $B\subset S$. If B be the minimal collection that contains all the open sets and is closed under unions and intersections both of length at most κ, then B is the class κ-Borel(S)*

Definition 1.7. *Let $S \in \{\mathbf{B}(\kappa), \mathbf{C}(\kappa)\}$.*

- *$X \subset S$ is a $\Sigma_1^1(\kappa)$ set if there is a set $Y \subset S \times S$ a closed set such that $pr(Y) = \{x \in S \mid \exists y \in S\ (x,y) \in Y\} = X$.*
- *$X \subset S$ is a $\Pi_1^1(\kappa)$ set if $S \backslash X$ is a $\Sigma_1^1(\kappa)$ set.*
- *$X \subset S$ is a $\Delta_1^1(\kappa)$ set if X is a $\Sigma_1^1(\kappa)$ set and a $\Pi_1^1(\kappa)$ set.*

Let $\theta \in \{2, \kappa\}$. A subset $T \subset \theta^{<\kappa}$ is a tree if for all $f \in T$ with $\alpha = dom(f) > 0$ and for all $\beta < \alpha$, $f \restriction \beta \in T$ and $f \restriction \beta < f$. In a similar way we can define trees on $\theta^{<\kappa} \times \theta^{<\kappa}$ and $\theta^{<\kappa} \times \theta^{<\kappa} \times \theta^{<\kappa}$ We say that a tree $T \subseteq \theta^{<\kappa}$ is pruned if for all $f \in T$ and $\beta > \alpha = dom(f)$, there is $g \in T$ such that $f = g \restriction \alpha$ and $\beta = dom(g)$. We define the body of a pruned tree T as the set

$$[T] = \{\eta \in \theta^\kappa \mid \forall \alpha < \kappa,\ \eta \restriction \alpha \in T\}.$$

Exercise 1.4. *Show that $A \subseteq \kappa^\kappa$ is closed if and only if there is a pruned tree of $\kappa^{<\kappa}$ such that $[T] = A$.*

A sequence $\langle \eta_i \mid< \gamma \rangle$ is a chain of length γ, if for all $i < j$, $\eta_i < \eta_j$.

Definition 1.8 (κ-*Borel**-set in $\mathbf{B}(\kappa), \mathbf{C}(\kappa)$)**.** *Let $S \in \{2^\kappa, \kappa^\kappa\}$.*

1. *A tree T is a κ^+, λ-tree if does not contain chains of length λ and its cardinality is less than κ^+. It is closed if every chain has a unique supremum in T.*
2. *A pair (T, h) is a κ-Borel*-code if T is a closed κ^+, λ-tree, $\lambda \leq \kappa$, and h is a function with domain T such that if $x \in T$ is a leaf, then $h(x)$ is a basic open set and otherwise $h(x) \in \{\cup, \cap\}$.*
3. *For an element $\eta \in S$ and a κ-Borel*-code (T, h), the κ-Borel*-game $B^*(T, h, \eta)$ is played as follows. There are two players, **I** and **II**. The game starts from the root of T. At each move, if the game is at node $x \in T$ and $h(x) = \cap$, then **I** chooses an immediate successor y of x and the game continues from this y. If $h(x) = \cup$, then **II** makes the choice. At limits the game continues from the (unique) supremum of the previous moves. Finally, if $h(x)$ is a basic open set, then the game ends, and **II** wins if and only if $\eta \in h(x)$.*
4. *A set $X \subseteq S$ is a κ-Borel*-set if there is a κ-Borel*-code (T, h) such that for all $\eta \in S$, $\eta \in X$ if and only if **II** has a winning strategy in the game $B^*(T, h, \eta)$.*

We will write $\mathbf{II} \uparrow B^*(T, h, \eta)$ when **II** has a winning strategy in the game $B^*(T, h, \eta)$.

Exercise 1.5. *Let $S \in \{2^\kappa, \kappa^\kappa\}$. We define κ-Borel**-sets in S by changing 2. in the previous definition for the following*

2′. *A pair (T, h) is a κ-Borel*-code if T is a closed κ^+, λ-tree, $\lambda \leq \kappa$, and h is a function with domain T such that if $x \in T$ is a leaf, then $h(x)$ is an open set and otherwise $h(x) \in \{\cup, \cap\}$.*

Show that $X \subseteq S$ is a κ-Borel**-set if and only if it is a κ-Borel*-set.

Recall that κ satisfies $\kappa^{<\kappa} = \kappa$, so it is regular. A set $X \subseteq \kappa$ is a club on κ if X is unbounded and any sequence $\langle \alpha_i \mid i < \gamma \rangle$ such that $\gamma < \kappa$ and for all $\alpha_i \in X$, satisfies $\bigcup_{i<\gamma} \alpha_i \in X$.

Exercise 1.6. *Show that the following set is an ideal:*

$$\{X \subseteq \kappa \mid \textit{exists a club } C \subseteq \kappa\ (X \cap C = \emptyset)\}.$$

Example 1.2. *Let $\mu < \kappa$ be a regular cardinal, we say that $X \subseteq \kappa$ is a μ-club if X is an unbounded set and it is closed under μ-limits.*

Let $\mu < \kappa$ be a regular cardinal. For all $\eta, \xi \in 2^\kappa$ we say that η and ξ are $=^2_\mu$ equivalent if the set $\{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\}$ contains a μ-club.

The relation $=^2_\omega$ is a κ-Borel* set. Let us define the following κ-Borel*-code (T, h):

- *$T = \{f \in \kappa^{<\omega+2} \mid f$ is strictly incresing$\}$.*
- *For f not a leave, $h(f) = \cup$ if $dom(f)$ is even and $h(f) = \cap$ if $dom(f)$ is odd.*
- *To define $h(f)$ for a leave f, first define the set $L(g) = \{f \in \kappa^{\omega+1} \mid g \subseteq f\}$ for all $g \in T$ with domain ω, and $\gamma_g = sup_{n<\omega}(g(n))$. Let $h \restriction L(g)$ be a bijection between $L(g)$ and the set $\{N_p \times N_q \mid p, q \in \kappa^{\gamma_g+1}, p(\gamma_g) = q(\gamma_g)\}$.*

Let us show that (T,h) codes $=^2_\omega$. Suppose $\eta =^2_\omega \xi$, so there is an ω-club C such that $\forall\alpha \in C\ \eta(\alpha) = \xi(\alpha)$. The following is a winning strategy for **II** *in the game $B^*(T, h, (\eta, \xi))$. For every even $n < \omega$, if the game is at f with $dom(f) = n$,* **II** *chooses an immediate successor f' of f, such that $f \subset f'$ and $f'(n) \in C$. Since C is closed under ω limits, after ω moves the game continues at $g \in \kappa^\omega$ strictly increasing with $\gamma = sup_{n<\omega}(g(n)) \in C$. So there is G an immediate successor of g, such that $h(G) = N_{\eta\restriction\gamma+1} \times N_{\xi\restriction\gamma+1}$. Finally if* **II** *chooses G in the ω move, then* **II** *wins.*

For the other direction, suppose $\eta \neq^2_\omega \xi$, so there is $A \subset S^\kappa_\omega$ stationary (S^κ_ω is the set of ω-cofinal ordinals below κ) such that for all $\alpha \in A$, $\eta(\alpha) \neq \xi(\alpha)$.

We will show that for every σ strategy of **II***, σ is not a winning strategy. Let σ be an strategy for* **II***, this mean that σ is a function from $\kappa^{<\omega+1} \to \kappa$. Notice that if* **II** *follows σ as a strategy, then when the game is at f, $dom(f) = n$ even,* **II** *chooses f' such that $f \subset f'$ and $f'(n) = \sigma((f(0), f(1), \dots, f(n-1)))$. Let C be the set of closed points of σ, $C = \{\alpha < \kappa \mid \sigma(\alpha^{<\omega}) \subseteq \alpha\}$, C is unbounded and closed under ω-limits. Therefore $C \cap A \neq \emptyset$. Let γ be the least element of $C \cap A$ that is an ω-limit of elements of C, and let $\{\gamma_n\}_{n<\omega}$ be a sequence of elements of C cofinal to γ. The following is a winning strategy for* **I** *in the game $B^*(T, h, (\eta, \xi))$, if* **II** *uses σ as an strategy.*

When the game is at f with $dom(f) = n$, n odd, then **I** *chooses an immediate successor f' of f, such that $f \subset f'$ and $f'(n)$ is the least element of $\{\gamma_n\}_{n<\omega}$ that is bigger than $f(n-1)$. This element always exists because $\{\gamma_n\}_{n<\omega}$ is cofinal to γ and $\gamma \in C$, γ is a closed point of σ. Since* **I** *is following σ as a strategy and γ is a closed point of σ, after ω moves the game continues at $g \in \kappa^\omega$ strictly increasing with $\gamma = sup_{n<\omega}(g(n)) \in C \cap A$. Since $\eta(\gamma) \neq \xi(\gamma)$, there is no G immediate successor of g, such that $(\eta, \xi) \in h(G)$. So it does not matter what* **II** *chooses in the ω move,* **I** *will win.*

The previous definitions are the generalization of the notions of Borel, Δ^1_1, and Borel* from descriptive set theory, the spaces ω^ω and 2^ω. A classical result in descriptive set theory states that the Borel class, the Δ^1_1 class, and the Borel* class are the same. This doesn't hold in generalized descriptive set theory as we will see.

Definition 1.9. *Let T be an tree without infinite branches. For all $t \in T$, we define $rk(t)$ as follows:*

- *If t is a leaf, then $rk(t) = 0$.*
- *If t is not a leaf, then $rk(t) = \cup\{rk(t') + 1 \mid t'^- = t\}$, where t'^- is the immediate predecessor of t'.*
- *If T is not empty and has a root, r, then the rank of T is denoted by $rk(T)$ and is equal to $rk(r)$.*

Exercise 1.7. *Show that if $A \subseteq \kappa^\kappa$ and $T = \{f \restriction \alpha : f \in A, \alpha < \kappa\}$, then $[T]$ is the closure of A.*

Exercise 1.8. *Show that if A and B are κ-Borel* sets, then $A \cup B$ and $A \cap B$ are κ-Borel* sets.*

Exercise 1.9. *Let (T, h) be a κ-Borel*-code. Show that if T is a κ^+,ω-tree, then for all η, $B^*(T, h, \eta)$ is determined, i.e.* **II** *has a winning strategy if and only if* **I** *doesn't have a winning strategy.*

Exercise 1.10.

1. *Prove Claim 1.11. (Hint: Use the previous exercise.)*
2. *Prove Claim 1.12.*

Theorem 1.10 (Friedman-Hyttinen-Weinstein(Kulikov), [5], Thm 17)**.** *κ-Borel $\subseteq$ κ-Borel**

Proof. Let us prove something even stronger. X is a κ-Borel set if and only if there is a κ-Borel*-code (T, h) such that (T, h) codes X and T is a κ^+,ω-tree.

We will show by induction over α that for every $X \in \Sigma^0_\alpha$, there is a κ-Borel*-code (T, h) such that (T, h) codes X and T is a κ^+,ω-tree.

For $\alpha = 1$. If $X \in \Sigma^0_\alpha$, then there is $\mathcal{B}$ a family of basic open sets such that $X = \bigcup \mathcal{B}$. Since $\kappa^{<\kappa} = \kappa$, $|\mathcal{B} = \kappa|$. So there is $\beta < \kappa$ such that $\mathcal{B} = \{B_i \mid i < \beta\}$. Let $T = \{\emptyset\} \cup \{(0, i) \mid i < \beta\}$, $h(\emptyset) = \cup$, and $h((0, i)) = B_i$, clearly this is a κ-Borel*-code that codes X.

Suppose α is such that for all $\beta < \alpha$ and $X \in \Sigma^0_\beta$, there is a κ-Borel*-code (T, h) such that (T, h) codes X and T is a κ^+,ω-tree.

Claim 1.11. *For all $\beta < \alpha$ and $X \in \Pi^0_\beta$, X is a κ-Borel* set.*

Suppose $X \in \Sigma^0_\alpha$, so $X = \bigcup_{\gamma<\kappa} A_\gamma$, where $A_\gamma \in \bigcup_{1\leq\beta<\alpha} \Pi^0_\beta$. By the previous claim we know that there are κ-Borel*-codes $\{(T_\gamma, h_\gamma)\}_{\gamma<\kappa}$ such that (T_γ, h_γ) codes A_γ and T_γ is a κ^+,ω-tree, for all γ. Let $\mathcal{T} = \{r\} \cup \bigcup_{\gamma<\kappa} T_\gamma \times \{\gamma\}$ be the tree ordered by $r < (x, j)$ for all $(x, j) \in \bigcup_{\gamma<\kappa} T_\gamma \times \{\gamma\}$, and $(x, \gamma) < (y, j)$ if and only if $\gamma = j$ and $x < y$ in T_γ. Let $T \subseteq \kappa^{<\omega}$ be a tree isomorphic to $\mathcal{T}$ and let $\mathcal{G} : T \to \mathcal{T}$ be a tree isomorphism. If $\mathcal{G}(x) \neq r$, then denote $\mathcal{G}(x)$ by $(\mathcal{G}_1(x), \mathcal{G}_2(x))$. Define h by $h(x) = \cup$ if $G(x) = r$, and $h(x) = h_{\mathcal{G}_2(x)}(\mathcal{G}_1(x))$.

Let us show that (T,h) codes X. Let $\eta \in X$, so there is $\gamma < \kappa$, such that $\eta \in X_\gamma$. **II** starts by choosing $\mathcal{G}^{-1}(x,\gamma)$, where x is the root of T_γ. **II** continues playing with the winning strategy from the game $B^*(T_\gamma,h_\gamma,\eta)$, choosing the element given by $\mathcal{G}^{-1}$. We conclude that $\mathbf{II}\uparrow B^*(T,h,\eta)$.

Let $\eta \notin X$, so for all $\gamma < \kappa$, $\eta \notin X_\gamma$, so **II** has no winning strategy for the game $B^*(T_\gamma,h_\gamma,\eta)$. Thus **II** cannot have a winning strategy for the game $B^*(T,h,\eta)$.

Let (T,h) be a κ-Borel*-code that codes X and T is a κ^+,ω-tree. We will use induction over the rank of T, to show that X is κ-Borel.

If $rk(T)=0$, then T has only one node r, thus $X=h(r)$ and X is a basic open set. Let $\alpha<\kappa^+$ be such that for all κ-Borel*-code (T',h') with T' κ^+,ω-tree and $rk(T')<\alpha$, (T',h') codes a κ-Borel set. If $rk(T)=\alpha$, then let $B=\{t\in T \mid t^-=r\}$, where r is the root of T. For all $t\in B$, define the code (T_t,h_t) as follows:

- $T_t=\{x\in T \mid t\leq x\}$,
- $h_t=h\restriction T_t$.

Since $rk(T)=\alpha$, for all $t\in B$, $rk(T_t)<\alpha$. By the induction hypothesis, (T_t,h_t) codes a κ-Borel set X_t.

Claim 1.12.
- *If $h(r)=\cup$, then $X=\cup_{t\in B}X_t$.*
- *If $h(r)=\cap$, then $X=\cap_{t\in B}X_t$.*

Since the class of κ-Borel sets is closed under unions and intersections of length κ, the proof follows from the previous claim. □

Theorem 1.13 (Friedman-Hyttinen-Weinstein(Kulikov), [5], Thm 17)**.**
1. *κ-Borel* $\subseteq \Sigma_1^1(\kappa)$.*
2. *κ-Borel $\subseteq \Sigma_1^1(\kappa)$.*
3. *κ-Borel $\subseteq \Delta_1^1(\kappa)$.*

Proof.
1. Let X be a κ-Borel* set, there is a κ-Borel* code (T,h) such that X is coded by (T,h).

   Since $\kappa^{<\kappa}=\kappa$, we can code the strategies $\sigma: T\to T$ by elements of κ^κ.

   Claim 1.14. *The set $Y=\{(\eta,\xi)\mid \xi$ is a code of a winning strategy for* **II** *in $B^*(T,h,\eta)\}$ is closed.*

   Proof. Let (η,ξ) be an element not in Y. So ξ is not a winning strategy for **II** in $B^*(T,h,\eta)\}$, there is $\alpha<\kappa$ such that for every $\zeta\in N_{\xi\restriction\alpha}$, ζ is not a winning strategy for **II** in $B^*(T,h,\eta)\}$. Otherwise T would have a branch of length κ. Because of the same reason, there is $\beta<\kappa$ such that for every $f\in N_{\eta\restriction\beta}$, $\zeta\in N_{\xi\restriction\alpha}$, ζ is not a winning strategy for **II** in $B^*(T,h,f)\}$. So $N_{\eta\restriction\beta}\times N_{\xi\restriction\alpha}$ is a subset of the complement of Y. □

   Since $pr(Y)=X$, we are done.
2. It follows from Theorem 1.10 and (1).
3. It follows from (2) and the fact that κ-Borel sets are closed under complement.

□

1.3 Separation theorem

Definition 1.15. *A dual of a κ-Borel* set B is the set $B^d=\{\eta\mid \mathbf{I}\uparrow B^*(T,h,\eta)\}$ where (T,h) satisfy $B=\{\eta\mid \mathbf{II}\uparrow B^*(T,h,\eta)\}$.*

Notice that the dual of a κ-Borel* set is not unique.

Definition 1.16. *If T is a tree on $\kappa^{<\kappa}\times\kappa^{<\kappa}$ and $f\in\kappa^\kappa$, let*

$$T(f)=\{g\restriction\alpha \mid (f\restriction\alpha,g\restriction\alpha)\in T\}.$$

Exercise 1.11. *Show that if $A\subseteq\kappa^\kappa$ is $\Pi_1^1(\kappa)$, then there is a tree T such that for all $f\in\kappa^\kappa$,*

$$f\in A\Leftrightarrow T(f) \text{ has no branch of length } \kappa.$$

Let us denote by TO the set of trees that don't have branches of length κ.

Definition 1.17.
- *Let T and S be trees. Then T is order preservingly embeddable into S, $T\leq S$, if there is a function $f: T\to S$ such that for all $t<_T t'$ implies $f(t)<_S f(t')$.*

- *If T is a tree, then σT is the tree of all initial segments of branches of T ordered by end-extension. We say that $T \ll T'$ if and only if $\sigma T \leq T'$.*

Definition 1.18. • *If A is a $\Pi^1_1(\kappa)$ set and T is a tree such that*

$$f \in A \Leftrightarrow T(f) \text{ has no branch of length } \kappa,$$

and $J \in TO$ we define $A^{T,J}$ as the set $\{f \in \kappa^\kappa \mid T(f) \leq J\}$.

- *If A is a $\Sigma^1_1(\kappa)$ set and T is a tree such that*

$$f \in A \Leftrightarrow T(f) \text{ has a branch of length } \kappa,$$

and $J \in TO$ we define $A_{T,J}$ as the set $\{f \in \kappa^\kappa \mid J \ll T(f)\}$.

Exercise 1.12. *1. Let A is a $\Pi^1_1(\kappa)$ set and T is a tree such that*

$$f \in A \Leftrightarrow T(f) \text{ has no branch of length } \kappa,$$

and $J \in TO$. Show that $A^{T,J} \subseteq A$.

2. Let A is a $\Sigma^1_1(\kappa)$ set and T is a tree such that

$$f \in A \Leftrightarrow T(f) \text{ has a branch of length } \kappa,$$

and $J \in TO$. Show that $A \subseteq A_{T,J}$.

Lemma 1.19 (Covering property, Mekler-Väänänen, [14], Proposition 11)**.** *Suppose A is a $\Pi^1_1(\kappa)$ set and T is a tree such that*

$$f \in A \Leftrightarrow T(f) \text{ has no branch of length } \kappa,$$

and $B \subseteq A$ is a $\Sigma^1_1(\kappa)$ set. The there is an element $J \in TO$ such that $B \subseteq A^{T,J}$.

Proof. Let S be a tree such that

$$f \in B \Leftrightarrow S(f) \text{ has a branch of length } \kappa.$$

Let T' be the set of triples $(f \restriction \alpha, g \restriction \alpha, h \restriction \alpha)$ such that $g \restriction \alpha \in T(f)$ and $h \restriction \alpha \in S(f)$. Notice that T' has no branch of length κ, otherwise $B \backslash A \neq \emptyset$.

Let $f \in B$ and let $\langle h \restriction \alpha \mid \alpha < \kappa \rangle$ be a branch in $S(f)$ of length κ. For $g \restriction \alpha \in T(f)$, let $\varrho : T(f) \to T'$ be defined as $\varrho(g \restriction \alpha) = (f \restriction \alpha, g \restriction \alpha, h \restriction \alpha)$. It is clear that ϱ is an order preserving embedding. Thus $f \in A^{T,T'}$. □

Lemma 1.20 (Mekler-Väänänen, [14], Proposition 32)**.** *Let T be a tree on $\kappa^{<\kappa} \times \kappa^{<\kappa}$ and J a tree with no branches of length κ. The sets*

$$B_0 = \{f \in \kappa^\kappa \mid T(f) \leq J\},$$
$$B_1 = \{f \in \kappa^\kappa \mid J \ll T(f)\}$$

are κ-Borel set and duals.*

Proof. Let H be the set of sequences $(\eta_0, (d_0, t_0), \eta_1, (d_1, t_1), \ldots, \eta_\delta, (d_\delta, t_\delta))$ satisfying the following:

- for all $\alpha \leq \delta$, $d_\alpha \in \{0, 1\}$.
- $d_\alpha = 1$ if and only if $\alpha = \delta$, $t_\delta = \emptyset$..
- $\langle t_\alpha \mid \alpha < \delta \rangle$ is a chain in J.
- For all $\alpha \leq \delta$, $\eta_\alpha \in \kappa^\alpha$, and $\langle \eta_\alpha \mid \alpha \leq \delta \rangle$ is a chain in $\kappa^{<\kappa}$.

Let K be the set of initial segments of the elements of H, ordered by end-extension (i.e. $x, y \in K$ are such that $x < y$ if and only if there is $\bar{a} \in H$ such that x, y are initial segments of $\bar{a}$ and x is an initial segment of y). notice that K is isomorphic to a κ^+, κ-tree. Thus we can construct a Borel*-code with K. Let us define $h : K \to \{\cup, \cap\} \cup \Sigma^0_1$, let $\bar{a} \in K$ be such that $\langle \eta \in \kappa^{<\kappa} \mid \eta \in \bar{a} \rangle$ has length δ

$$h(\bar{a}) = \begin{cases} \cup & \text{if } \bar{a} \text{ ends with } \eta_\alpha \in \kappa^{<\kappa}, \\ \cap & \text{if } \bar{a} \text{ ends with } (d_\alpha, t_\alpha) \text{ and } d_\alpha = 0 \text{ or } \bar{a} = \langle\rangle, \\ \{f \in \kappa^\kappa \mid (f \restriction \delta, \eta_\delta) \notin T\} & \text{otherwise.} \end{cases}$$

Claim 1.21. *1.*

$$T(f) \leq J \Leftrightarrow \textbf{II} \textit{ has a winning strategy for } B^*(K, h, f).$$

2.

$$J \ll T(f) \Leftrightarrow \textbf{I} \textit{ has a winning strategy for } B^*(K, h, f).$$

Proof. 1. Let us suppose that $T(f) \leq J$ and $G : T(f) \to J$ witnesses it. Let us define the following strategy for **II**, if $(f \restriction \delta, \eta_\delta) \notin T$, **II** chooses $(1, \emptyset)$. Otherwise, $\eta_\delta \in T(f)$, and **II** chooses $(0, G(\eta_\delta))$. It is clear that this is a winning strategy for **II**. For the other direction, let ρ be a winning strategy for **II**. When the game is at $\bar{a}$ ending in η_α and the strategy ρ says that **II** has choose $(0, t_\alpha)$, then $\eta_\alpha \in T(F)$, so $G(\eta_\alpha) = t_\alpha$ is an embedding.

2. Let us suppose $J \ll T(f)$ and $G : \sigma J \to T(f)$ witnesses it. Let us define the following strategy for **I**, suppose the game is at $\bar{a}$ ending with $(0, t_\alpha)$, so $\langle t_\beta \mid \beta < \alpha \rangle$ is a chain in J. Thus **I** should choose $G(\langle t_\beta \mid \beta \leq \alpha \rangle)$. It clear that this is a winning strategy for **I**. The other direction is similar as in the previous item.

□

□

Theorem 1.22 (Separation property, Mekler-Väänänen, [14], Corollary 34)**.** *Suppose A and B are disjoint $\Sigma_1^1(\kappa)$ sets. There are κ-Borel* sets C_0 and C_1 such that $A \subseteq C_0$, $B \subseteq C_1$, and C_0 and C_1 are duals.*

Proof. Since B is $\Sigma_1^1(\kappa)$, $\kappa^\kappa \backslash B$ is $\Pi_1^1(\kappa)$ and there is T a tree such that

$$f \in \kappa^\kappa \backslash B \Leftrightarrow T(f) \text{ has no branch of length } \kappa,$$

and $A \subseteq \kappa^\kappa \backslash B$. Thus by the covering property, there is $J \in TO$ such that $A \subseteq (\kappa^\kappa \backslash B)^{T,J}$. By the previous exercise, $B \subseteq B_{T,J}$. From Definition 1.18

$$(\kappa^\kappa \backslash B)^{T,J} = \{f \in \kappa^\kappa \mid T(f) \leq J\},$$

$$B_{T,J} = \{f \in \kappa^\kappa \mid J \ll T(f)\}.$$

The proof follows from Lemma 1.20. □

Theorem 1.23 (Friedman-Hyttinen-Weinstein(Kulikov), [5], Theorem 17)**.** *$\Delta_1^1(\kappa) \subseteq \kappa$-Borel**

Proof. Let A be a $\Delta_1^1(\kappa)$ set. Let $B = \mathbf{B}(\kappa) \backslash A$, by Theorem 1.22, there are κ-Borel* sets C_0 and C_1 such that $A \subseteq C_0$, $B \subseteq C_1$, and C_0 and C_1 are duals. Since C_0 and C_1 are duals, C_0 and C_1 are disjoint. So $C_0 \cap B = \emptyset$, then $A = C_0$, $B = C_1$. □

Corollary 1.24 (Mekler-Väänänen, [14], Corollary 35)**.** *X is $\Delta_1^1(\kappa)$ if there is a κ-Borel*-code (T, h) that codes X and*

$$\textbf{II} \uparrow B^*(T, h, \eta) \Leftrightarrow \textbf{I} \not\uparrow B^*(T, h, \eta)$$

for all $\eta \in \kappa^\kappa$ the game is determined.

Exercise 1.13. *Prove the claims of the following proof.*

Theorem 1.25 (Friedman-Hyttinen-Weinstein(Kulikov), [5], Theorem 18)**.** *1. κ-Borel$\subsetneq \Delta_1^1(\kappa)$*

2. $\Delta_1^1(\kappa) \subsetneq \Sigma_1^1(\kappa)$

Proof. 1. Let $\xi \mapsto (T_\xi, h_\xi)$ be a continuous coding of the κ-Borel*-codes with T a $\kappa^+\omega$-tree, such that for all $\kappa^+\omega$-tree, T, and h, there is ξ such that $(T_\xi, h_\xi) = (T, h)$.

Claim 1.26. *The set $B = \{(\eta, \xi) \mid \eta$ is in the set coded by $(T_\xi, h_\xi)\}$ is $\Delta_1^1(\kappa)$ and is not κ-Borel, otherwise $D = \{\eta \mid (\eta, \eta) \notin B\}$ would be Borel.*

*(Hint: use the set $C = \{(\eta, \xi, \sigma) \mid \sigma$ is a winning strategy for **II** in $B^*(T_\xi, h_\xi, \eta)\}$).*

2.

Claim 1.27. *There is $A \subseteq 2^\kappa \times 2^\kappa$ such that if $B \subseteq 2^\kappa$ is a $\Sigma_1^1(\kappa)$ set, then there is $\eta \in 2^\kappa$ such that $B = \{\xi \mid (\xi, \eta) \in A\}$.*

(Hint: the construction used in the classical case works too).

The set $D = \{\eta \mid (\eta, \eta) \in A\}$ is $\Sigma_1^1(\kappa)$ but not $\Pi_1^1(\kappa)$.

□

From the previous results, we can see that

$$\kappa\text{-Borel} \subsetneq \Delta_1^1(\kappa) \subsetneq \Sigma_1^1(\kappa)$$

and

$$\Delta_1^1(\kappa) \subseteq \kappa\text{-Borel}^* \subseteq \Sigma_1^1(\kappa).$$

Therefore we are missing to determine whether one of the following holds:

- $\Delta_1^1(\kappa) \subsetneq \kappa\text{-Borel}^* \subsetneq \Sigma_1^1(\kappa)$;
- $\Delta_1^1(\kappa) \subsetneq \kappa\text{-Borel}^* = \Sigma_1^1(\kappa)$;
- $\Delta_1^1(\kappa) = \kappa\text{-Borel}^* \subsetneq \Sigma_1^1(\kappa)$.

As we will see, only case has not been answered.

Question 1.28. *Is the following consistent $\Delta_1^1(\kappa) = \kappa$-Borel$^* \subsetneq \Sigma_1^1(\kappa)$?*

2 Reductions

Let $\beta, \theta \in \{2, \kappa\}$, and E_1 and E_2 be equivalence relations on β^κ and θ^κ, respectively. We say that E_1 *is reducible to* E_2 if there is a function $f\colon \beta^\kappa \to \theta^\kappa$ that satisfies

$$(\eta, \xi) \in E_1 \iff (f(\eta), f(\xi)) \in E_2.$$

We call f a *reduction* of E_1 to E_2 and we denote by $E_1 \hookrightarrow_r E_2$ the existence of a reduction of E_1 to E_2. It is clear that $E_1 \hookrightarrow_r E_2$ holds if and only if E_1 doesn't have more equivalence classes than E_2.

Definition 2.1 (Reductions). *Apart from a "cardinality" reduction, $\hookrightarrow_r$, we define the following notions which allow us to have a better spectrum of complexities.*

- **Borel reduction.** *A function $f\colon \beta^\kappa \to \theta^\kappa$ is said to be* κ-Borel *if for any open set $A \subseteq \theta^\kappa$, $f^{-1}[A]$ is a κ-Borel set. The existence of a κ-Borel reduction of E_0 to E_1 is denoted by $E_0 \hookrightarrow_B E_1$.*
- **Continuous reduction.** *The existence of a continuous reduction of E_0 to E_1 is denoted by $E_0 \hookrightarrow_c E_1$.*
- **Lipschitz reduction.** *For all $\eta, \xi \in \beta^\kappa$, denote*

  $$\Delta(\eta, \xi) := \min(\{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cup \{\kappa\}).$$

  A function $f\colon \beta^\kappa \to \theta^\kappa$ is said to be Lipschitz *if for all $\eta, \xi \in \beta^\kappa$,*

  $$\Delta(\eta, \xi) \leq \Delta(f(\eta), f(\xi)).$$

  The existence of a Lipschitz reduction of E_0 to E_1 is denoted by $E_0 \hookrightarrow_L E_1$.

2.1 Basic reductions

Fact 2.2 (Folklore). *If $f\colon \kappa^\kappa \to \kappa^\kappa \times \kappa^\kappa$ is a continuous functions, then for all κ-Borel $X \subseteq \kappa^\kappa \times \kappa^\kappa$, $f^{-1}[X]$ is κ-Borel.*

Proof. Let us proceed by induction over Σ_α^0. Since f is continuous, if $X \in \Sigma_1^0$, then $f^{-1}[X]$ is open. Thus X is κ-Borel. Let us suppose that $\alpha < \kappa^+$ is such that for all $\beta < \alpha$, if $X \in \Sigma_\beta^0$, then $f^{-1}[X]$ is κ-Borel. Let $X \in \Pi_\beta^0$, for some $\beta < \alpha$. Then, $X = \kappa^\kappa \backslash A$, for some $A \in \Sigma_\beta^0$. It is clear that $f^{-1}[X] = \kappa^\kappa \backslash f^{-1}[A]$. By the induction hypothesis $f^{-1}[A]$ is κ-Borel, so $f^{-1}[X]$ is κ-Borel.

Let $X \in \Sigma_\alpha^0$. So, $X = \bigcup_{\gamma<\kappa} A_\gamma$, where $A_\gamma \in \bigcup_{\beta<\alpha} \Pi_\beta^0$. It is easy to see that $f^{-1}[X] = \bigcup_{\gamma<\kappa} f^{-1}[A_\gamma]$. As it was proved above, A_γ is κ-Borel, therefore X is κ-Borel. □

Exercise 2.1. *Show that if $f : \kappa^\kappa \to \kappa^\kappa$ is a κ-Borel function, then for all κ-Borel* set $B \subseteq \kappa^\kappa$, $f^{-1}[B]$ is a κ-Borel* set.*

Fact 2.3 (Folklore). *Suppose $E_0 \hookrightarrow_r E_1$. Then the following hold:*

- *If E_1 is κ-Borel and $E_0 \hookrightarrow_B E_1$, then E_0 is κ-Borel.*
- *If E_1 is $\Delta_1^1(\kappa)$ and $E_0 \hookrightarrow_B E_1$, then E_0 is $\Delta_1^1(\kappa)$.*

- *If E_1 is open and $E_0 \hookrightarrow_c E_1$, then E_0 is open.*

Proof. It follows from the previous exercise and the following claim.

Claim 2.4. *$\kappa^\kappa \times \kappa^\kappa$ and κ^κ are homeomorphic.*

Proof. Let $g : \kappa \to \{0,1\} \times \kappa$ be a bijection, we denote $g(\alpha)$ by $(g_1(\alpha), g_2(\alpha))$. Let us define $F : \kappa^\kappa \times \kappa^\kappa \to \kappa^\kappa$ by $F((\eta_0, \eta_1))(\alpha) = h(\alpha) = \eta_{g_1(\alpha)}(g_2(\alpha))$. Let us show that F is a homeomorphism.

Injective. Let us assume, towars contradiction, that there are (η_0, η_1) and (ξ_0, ξ_1) such that $F((\eta_0, \eta_1)) = F((\xi_0, \xi_1))$. Thus, for all $\alpha < \kappa$, $\eta_{g_1(\alpha)}(g_2(\alpha)) = \xi_{g_1(\alpha)}(g_2(\alpha))$. Let $A_0 = \{\alpha < \kappa \mid g_1(\alpha) = 0\}$ and $A_1 = \{\alpha < \kappa \mid g_1(\alpha) = 1\}$. Therefore, for all $\alpha \in A_0$, $\eta_0(g_2(\alpha)) = \xi_0(g_2(\alpha))$ and for all $\alpha \in A_1$, $\eta_1(g_2(\alpha)) = \xi_1(g_2(\alpha))$. Finally, since g is a bijection, $g_2[A_0] = g_2[A_1] = \kappa$, for all $\beta < \kappa$, $\eta_0(\beta) = \xi_0(\beta)$ and $\eta_1(\beta) = \xi_1(\beta)$. a contradiction.

Surjective. Let A_0 and A_1 as before. Let $\eta \in \kappa^\kappa$. Let us define ξ_0 by $\xi_0(g_2(\alpha)) = \eta(\alpha)$ for all $\alpha \in A_0$. Let us define ξ_1 by $\xi_1(g_2(\alpha)) = \eta(\alpha)$ for all $\alpha \in A_1$. Clearly $F((\xi_1, \xi_0)) = \eta$.

Continuity. Let $\alpha < \kappa$, and η, ξ_0 and ξ_1 be such that $(\xi_0, \xi_1) \in F^{-1}[N_{\eta\restriction\alpha}]$. So, for all $\beta < \alpha$, $\eta(\beta) = F(\xi_0, \xi_1)(\beta) = \xi_{g_1(\beta)}(g_2(\beta))$. Let $\gamma = sup\{g_2(\beta) \mid \beta < \alpha\}$ and $(\zeta_0, \zeta_1) \in N_{\xi_0\restriction\gamma} \times N_{\xi_1\restriction\gamma}$. Clearly for all $\beta < \alpha$, $F((\zeta_0, \zeta_1))(\beta) = \zeta_{g_1(\beta)}(g_2(\beta)) = \xi_{g_1(\beta)}(g_2(\beta)) = F((\xi_0, \xi_1))(\beta) = \eta(\beta)$. Thus $N_{\xi_0\restriction\gamma} \times N_{\xi_1\restriction\gamma} \subseteq F^{-1}[N_{\eta\restriction\alpha}]$.

Open sets. Let $\alpha_0, \alpha_1 < \kappa$, and η, ξ_0 and ξ_1 be such that $\eta \in F[N_{\xi_0\restriction\alpha_0} \times N_{\xi_1\restriction\alpha_1}]$. Let $\gamma = sup\{g_2^{-1}(x, \beta) \mid x \in \{0,1\} \ \& \ \beta < \max(\alpha_1, \alpha_2)\}$, $\zeta \in N_{\eta\restriction\gamma}$, and ϑ_0 and ϑ_1 be such that $F((\vartheta_0, \vartheta_1)) = \zeta$, thus for all $\beta < \gamma$, $F((\vartheta_0, \vartheta_1))(\beta) = \nu_{g_1(\beta)}(g_2(\beta)) = \zeta(\beta) = \eta(\beta)$. We conclude that $N_{\eta\restriction\gamma} \in F[N_{\xi_0\restriction\alpha_0} \times N_{\xi_1\restriction\alpha_1}]$. □

If $E_0 \hookrightarrow_B E_1$, then we would have $[f \times f]^{-1}[E_1] = E_0$ and since E_1 is Borel*, this yield E_0 to be Borel*. □

Fact 2.5 (Folklore)**.** *Let E be a κ-Borel equivalence relation. Then the equivalence classes of E are κ-Borel.*

Proof. Let $x \in \kappa^\kappa$, and let us define $f : \kappa^\kappa \to \kappa^\kappa \times \kappa^\kappa$ as $f(\eta) = (\eta, x)$. It is clear that f is continuous. On the other hand $[x]_E$ (the E-equivalence class of x) is equal to $f^{-1}[(\kappa^\kappa \times \{x\}) \cap E]$. Clearly $\kappa^\kappa \times \{x\}$ is κ-Borel and since E is κ-Borel, by Fact 2.2 $f^{-1}[(\kappa^\kappa \times \{x\}) \cap E]$ is κ-Borel. □

Lemma 2.6 (Mangraviti-Motto Ros, [13])**.** *Let E_1 be a κ-Borel equivalence relation with $\gamma \leq \kappa$ equivalence classes and E_2 be an equivalence relation with θ equivalence classes. If $\gamma \leq \theta$, then $E_1 \hookrightarrow_B E_2$.*

Proof. Let us choose $\langle y_i \mid i < \gamma \rangle$ representatives of each E_1-equivalence class and $\langle x_i \mid i < \theta \rangle$ representatives of each E_2-equivalence class. Let us define $F : \kappa^\kappa \to \kappa^\kappa$ as $F(\eta) = x_i$, where $i < \gamma$ is such that $\eta \ E_1 \ y_i$. Since $\gamma \leq \theta$, F is well defined.

Claim 2.7. *$\eta \ E_1 \ \xi$ if and only if $F(\eta) \ E_2 \ F(\xi)$.*

Proof. By the way F was defined, it is enough to prove that $\eta \ E_1 \ \xi$ if and only if $x_i \ E_2 \ x_j$, where i and j are such that $\eta \ E_1 \ y_i$ and $\xi \ E_1 \ y_j$. Since E_1 is an equivalence relation, $\eta \ E_1 \ \xi$ if and only if $y_i \ E_1 \ y_j$.

If $\eta \ E_1 \ \xi$, then $y_i \ E_1 \ y_j$ and $i = j$. We conclude that $x_i = x_j$ and $x_i \ E_2 \ x_j$. The other direction is similar. □

Let us show that F is κ-Borel. Let $X \subseteq \kappa^\kappa$ be an open set. Then,

$$F^{-1}[X] = \bigcup_{x_i \in X} [y_i]_{E_1}.$$

By the previous fact, $[y_i]_{E_1}$ is κ-Borel for all $i < \gamma$. Since $\gamma \leq \kappa$, $\bigcup_{x_i \in X} [y_i]_{E_1}$ is κ-Borel. □

Definition 2.8 (Counting classes)**.** *Let $0 < \varrho \leq \kappa$ be a cardinal. Let us define the equivalence relation $0_\varrho \in \kappa^\kappa \times \kappa^\kappa$ as follows: $\eta \ 0_\varrho \ \xi$ if and only if one of the following holds:*

- *ϱ is finite:*
  - *$\eta(0) = \xi(0) < \varrho - 1$;*
  - *$\eta(0), \xi(0) \geq \varrho - 1$.*
- *ϱ is infinite:*
  - *$\eta(0) = \xi(0) < \varrho$;*
  - *$\eta(0), \xi(0) \geq \varrho$.*

Lemma 2.9 (Moreno, [16])**.** *Let E be a Borel equivalence relation with $\varrho \leq \kappa$ equivalence classes. Then*

$$E \hookrightarrow_B 0_\varrho \text{ and } 0_\varrho \hookrightarrow_L E.$$

If E is not open, then $E \not\hookrightarrow_c 0_\varrho$.

Proof. It is clear that for all 0_ϱ is open, then by Lemma 2.6, $E \hookrightarrow_B 0_\varrho$.

Let show the case $\varrho \geq \omega$, let $\langle x_i \mid i \leq \varrho \rangle$ representatives of each E-equivalence class. Clearly the function

$$F(\eta) = \begin{cases} x_{\eta(0)+1} & \text{if } \eta(0) < \varrho, \\ x_0 & \eta(0) \geq \varrho. \end{cases}$$

is Lipschitz and a reduction from 0_ϱ to E, i.e. $0_\varrho \hookrightarrow_L E$.

Finally, suppose $E \hookrightarrow_c 0_\varrho$. Since 0_ϱ is open, by Fact 2.3, E is open. □

Let us define $E_0^{<\kappa}$, the *equivalence modulo bounded*, as:

$$E_0^{<\kappa} := \{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid \exists \alpha < \kappa\ [\forall \beta > \alpha\ (\eta(\beta) = \xi(\beta)])\}.$$

Let id_2 be the identity relation of 2^κ.

Exercise 2.2. *Show that $E_0^{<\kappa}$ is an equivalence relation.*

Theorem 2.10 (Friedman-Hyttinen-Weisnstein(Kulikov), [5] Theorem 34)**.** *1. $E_0^{<\kappa}$ is κ-Borel.*

2. $id_2 \hookrightarrow_c E_0^{<\kappa}$.

Proof. 1. Let us denote by $[\kappa]^{<\kappa}$ the set of subsets of κ of size smaller than κ. Clearly

$$E_0^{<\kappa} = \bigcup_{A \in [\kappa]^{<\kappa}} \bigcap_{\alpha \notin A} \{(\eta, \xi) \mid \eta(\alpha) = \xi(\alpha)\}$$

and $\{(\eta, \xi) \mid \eta(\alpha) = \xi(\alpha)\}$ is open.

2. Let $(A_i)_{i<\kappa}$ be a partition of κ such that for all $i < \kappa$, $|A_i| = \kappa$. Let us define $F : 2^\kappa \to \kappa^\kappa$ by $F(\eta)(\alpha) = \eta(i)$if and only if $\alpha \in A_i$. Clearly, if $\eta = \xi$, then $F(\eta) = F(\xi)$ and $F(\eta)\ E_0^{<\kappa}\ F(\xi)$. If $\eta \neq \xi$, then there is $i < \kappa$ such that $\eta(i) \neq \xi(i)$. So

$$A_i \subseteq \{\alpha < \kappa \mid F(\eta)(\alpha) \neq F(\xi)(\alpha)\}.$$

Since $|A_i| = \kappa$, we conclude that $F(\eta)$ and $F(\xi)$ are not $E_0^{<\kappa}$ equivalent.

□

Definition 2.11. *Let $S \subseteq \kappa$ be an unbounded set. We say that a function $f : \kappa^\kappa \to \kappa^\kappa$ is S-recursive if there is a function $H : \kappa^{<\kappa} \to \kappa^{<\kappa}$ such that for all $\alpha \in S$ and $\eta \in \kappa^\kappa$, $f(\eta)(\theta) = H(\eta \restriction \alpha)(\theta)$ for all $\theta < min(S \backslash (\alpha+1))$.*

Exercise 2.3 (Moreno, [16])**.** *Let $S \subseteq \kappa$ be unbounded and $f : \kappa^\kappa \to \kappa^\kappa$ an S-recursive function.*

1. *f is continuous.*
2. *If S is a club that satisfies the following:*

   (†) $\alpha_m = min(S)$ is such that for all $\eta, \xi \in \kappa^\kappa$ and $\beta < \alpha_m$, $\eta \restriction \beta = \xi \restriction \beta$ implies $f(\eta) \restriction \beta = f(\xi) \restriction \beta$.

   Then f is Lipschitz.

Exercise 2.4 (Moreno, [16])**.** *1. Find $S \subseteq \kappa$ and a function f, such that f is S-recursive but not κ-recursive.*

2. Find $S \subseteq \kappa$ and a function f, such that f is κ-recursive but not S-recursive.

2.2 Equivalence modulo S

Definition 2.12. *We say that a set $S \subseteq \kappa$ is stationary if for all club $C \subseteq \kappa$, $S \cap C \neq \emptyset$.*

Notice that if $S \subseteq \kappa$ is stationary and $C \subseteq \kappa$ is a club, then $S \cap C$ is stationary.

Definition 2.13. *Given $S \subseteq \kappa$ and $\theta \in \{2, \kappa\}$, we define the equivalence relation $=^\theta_S \subseteq \theta^\kappa \times \theta^\kappa$, as follows*

$$\eta =^\theta_S \xi \iff \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S \text{ is non-stationary.}$$

It is clear that $=^\theta_S \neq \theta^\kappa \times \theta^\kappa$ if and only if S is stationary.

Exercise 2.5. *Show that $\eta =^\theta_S \xi$ if and only if there is a club $C \subseteq \kappa$, such that $C \cap S \subseteq \{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\}$.*

Exercise 2.6. *Show that if C is a club, then the set of limits of C is also a club.*

Exercise 2.7. *Prove Lemma 2.14*

Lemma 2.14 (Monotonicity, Fernandes-Moreno-Rinot, [3] Lemma 2.7)**.** *Suppose $\theta, \theta', \lambda, \lambda' \in \{2, \kappa\}$ are such that $\theta \leq \theta'$, $\lambda \leq \lambda'$, and, $X \subseteq X'$ and $S \subseteq S'$ are stationary sets such that $=^{\theta'}_{X'} \hookrightarrow_c =^{\lambda}_{S}$, then $=^{\theta}_{X} \hookrightarrow_c =^{\lambda'}_{S'}$.*

Definition 2.15. *Let (T, h) be a κ-Borel*-code and $\alpha < \kappa$. Let $(T_\alpha, h_\alpha) = (T, h) \restriction \alpha$ be the α-approximation of (T, h) defined by $T_\alpha = T \cap \alpha^{<\omega}$ and $h_\alpha = h \restriction T_\alpha$.*

We say that a κ-Borel equivalence relation $E \subseteq 2^\kappa \times 2^\kappa$ has an approximation if there is a κ-Borel*-code, (T, h), such that the following hold

- T doesn't have infinite branches,
- (T, h) codes E,
- there is a club C such that for all $\alpha \in C$, $(T, h) \restriction \alpha$ codes an equivalence relation E_α,
- for all $\alpha \in C$ and leaf $l \in T \cap \alpha^{<\omega}$, there are $\eta, \xi \in 2^{<\alpha}$ such that $h_\alpha(l) = N_\eta \times N_\xi$.

Lemma 2.16 (Friedman-Hyttinen-Weisnstein(Kulikov), [4] Theorem 11)**.** *Let E be a κ-Borel equivalence relation with an approximation (T, h) and $C \subseteq \kappa$. For all stationary set $S \subseteq \kappa$, $E \hookrightarrow_c =^{\kappa}_{S}$.*

Proof. Since E is approximated by (T, h) and $C \subseteq \kappa$, $(T, h) \restriction \alpha$ is an equivalence relation for all $\alpha \in C$. Let us denote these equivalence relations by E_α. For all $\alpha \in C$, let $\langle x^\alpha_i \mid 0 < i < \kappa \rangle$ be an enumeration of the E_α-equivalence classes. Let us define the function $F : \kappa^\kappa \to \kappa^\kappa$ by

$$F(\eta)(\alpha) = \begin{cases} i & \text{if } \alpha \in C \text{ and } \eta \in x^\alpha_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that $\eta \; E \; \xi$ if and only if $F(\eta) \; =^{\kappa}_{S} \; F(\xi)$.

If $\eta \; E \; \xi$, then **II** has a winning strategy σ for the game $B^*(T, h, (\eta, \xi))$. Notice that the set $D = \{\alpha < \kappa \mid \sigma(\alpha^{<\omega}) \subseteq \alpha\}$ is a club, thus for all $\alpha \in C \cap D$, σ is a winning strategy of **II** for the game $B^*(T_\alpha, h_\alpha, (\eta, \xi))$. We conclude that $\eta \; E_\alpha \xi$ and $F(\eta)(\alpha) = F(\xi)(\alpha)$. We conclude that $C \cap D \subseteq \{\alpha < \kappa \mid F(\eta)(\alpha) = F(\xi)(\alpha)\}$ and $\{\alpha < \kappa \mid F(\eta)(\alpha) \neq F(\xi)(\alpha)\} \cap S$ is non-stationary. So $F(\eta) \; =^{\kappa}_{S} \; F(\xi)$.

From Exercise 1.9 and a similar argument, it is possible to show that there is a club $D \subseteq \kappa$ such that $C \cap D \subseteq \{\alpha < \kappa \mid F(\eta)(\alpha) \neq F(\xi)(\alpha)\}$. Thus $\{\alpha < \kappa \mid F(\eta)(\alpha) \neq F(\xi)(\alpha)\} \cap S$ is stationary. So $F(\eta) \; \neq^{\kappa}_{S} \; F(\xi)$.

Exercise 2.8. *Show that F is C-recursive and continuous.*

□

Exercise 2.9. *Prove Lemma 2.17.*

Lemma 2.17 (Fernandes-Moreno-Rinot, [3] Lemma 2.10)**.** *Suppose κ is such that $|\kappa| = |2^\lambda|$ for some $\lambda < \kappa$, and $X, S \subseteq \kappa$ be stationary sets. Show that if $=^2_X \hookrightarrow_c =^2_S$, then $=^\kappa_X \hookrightarrow_c =^\kappa_S$.*

(Hint: Similar to Fact 2.10 (2).) Use the following two facts:

- *If $\langle D_i \mid i < \gamma < \kappa \rangle$ is a sequence of clubs of κ, then $\bigcap_{i<\gamma} D_i$ is a club of κ.*
- *If $S \subseteq \kappa$ is stationary and $\langle S_i \mid i < \gamma < \kappa \rangle$ is a sequence of disjoint subsets of S such that $\bigcup_{i<\gamma} S_i = S$, then there is $j < \gamma$, such that S_j is a stationary set of κ.*

Show that the following function F is a reduction:

- *Let $h : \kappa \to 2^\lambda$ is a bijection.*
- *Define $\pi : \kappa^\kappa \to (2^\kappa)^\lambda$ by $\pi(\eta) = \langle \eta_i \mid i < \lambda \rangle$ where*

$$\eta_i(\alpha) = h(\eta(\alpha))(i).$$

- *Let $f : 2^\kappa \to 2^\kappa$ a continuous reduction from $=^2_X$ to $=^2_S$.*
- *Define $F : \kappa^\kappa \to \kappa^\kappa$ by $F(\eta) = \zeta$, where $\pi(\eta) = \langle \eta_i \mid i < \lambda \rangle$ and $\pi(\zeta) = \langle f(\eta_i) \mid i < \lambda \rangle$.*

2.3 The approximation lemma

Definition 2.18 (S-approximation)**.** *Let $\theta \in \{2, \kappa\}$ and let $S \subseteq \kappa$ be a stationary set, we say that an equivalence relation $E \subseteq \theta^\kappa \times \theta^\kappa$ has an S-approximation if there is $\langle E_\alpha \mid \alpha < \kappa \rangle$ a sequence of relations, $E_\alpha \subseteq \theta^\alpha \times \theta^\alpha$, such that the following hold:*

1. *There is $C \subseteq \kappa$ a club such that for all $\alpha \in C$, E_α is an equivalence relation.*
2. *For all $\eta, \xi \in \theta^\kappa$, if $\eta \; E \; \xi$, then there is $D \subseteq C$ a club, such that for all $\alpha \in D$,*
$$\eta \restriction \alpha \; E_\alpha \; \xi \restriction \alpha.$$
3. *For all $\eta, \xi \in \theta^\kappa$, if $\neg(\eta \; E \; \xi)$, then there is $S' \subseteq S$ a stationary set, such that for all $\alpha \in S'$,*
$$\neg(\eta \restriction \alpha \; E_\alpha \; \xi \restriction \alpha).$$

Lemma 2.19 (Approximation lemma in κ^κ)**.** *Suppose $\theta \in \{2, \kappa\}$, $S \subseteq \kappa$ is a stationary set, and $E \subseteq \theta^\kappa \times \theta^\kappa$ is an equivalence relation with an S-approximation, $\langle E_\alpha \mid \alpha < \kappa \rangle$. Then*
$$E \hookrightarrow_L =^\kappa_S .$$

Proof. Let $C \subseteq \kappa$ be the club that witnesses that $\langle E_\alpha \mid \alpha < \kappa \rangle$ is an S-approximation. For all $\alpha \in C$, let $\langle x_i^\alpha \mid 0 < i < \kappa \rangle$ be an enumeration of the E_α-equivalence classes (this can be done since $\kappa^{<\kappa} = \kappa$). Let us define $F : \theta^\kappa \to \kappa^\kappa$ as follows:
$$F(\eta)(\alpha) = \begin{cases} i & \text{if } \alpha \in C \text{ and } \eta \restriction \alpha \in x_i^\alpha, \\ 0 & \text{otherwise.} \end{cases}$$
Let us show that $\eta \; E \; \xi$ if and only if $F(\eta) \; =^\kappa_S \; F(\xi)$.

Claim 2.20. *$\eta \; E \; \xi$ implies $F(\eta) \; =^\kappa_S \; F(\xi)$.*

Proof. Suppose $\eta, \xi \in \theta^\kappa$ are such that $\eta \; E \; \xi$. Since $\langle E_\alpha \mid \alpha < \kappa \rangle$ is an S-approximation, by Definition 2.18 item 2, there is a club $D \subseteq C$ such that for all $\alpha \in D$,
$$\eta \restriction \alpha \; E_\alpha \; \xi \restriction \alpha.$$
So, for all $\alpha \in D \cap S$, $F(\eta)(\alpha) = F(\xi)(\alpha)$. Thus $\{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S$ is non-stationary and we conclude that $F(\eta) \; =^\kappa_S \; F(\xi)$. ☐

Claim 2.21. *$\neg(\eta \; E \; \xi)$ implies $\neg(F(\eta) \; =^\kappa_S \; F(\xi))$.*

Proof. Suppose $\eta, \xi \in \theta^\kappa$ are such that $\neg(\eta \; E \; \xi)$. Since $\langle E_\alpha \mid \alpha < \kappa \rangle$ is an S-approximation, by Definition 2.18 item 3, there is a stationary subset $S' \subseteq S$ such that for all $\alpha \in S'$,
$$\neg(\eta \restriction \alpha \; E_\alpha \; \xi \restriction \alpha).$$
So, for all $\alpha \in C \cap S'$, $F(\eta)(\alpha) \neq F(\xi)(\alpha)$. Thus $C \cap S' \subseteq \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S$ is stationary and we conclude that $\neg(F(\eta) \; =^\kappa_S \; F(\xi))$. ☐

Claim 2.22. *F is C-recursive*

Proof. Let us define $H : \theta^{<\kappa} \to \kappa^{<\kappa}$ as follows:
$$H(\eta \restriction \alpha) = \begin{cases} F(\eta) \restriction \alpha' & \text{if } \alpha \in C \text{ and } \alpha' = min(C \backslash (\alpha + 1)), \\ \bar{0}_\alpha & \text{otherwise.} \end{cases}$$
Where $\bar{0}_\alpha$ is the function constant to 0 with domain α. Clearly, if $\alpha, \beta \in C$ are such that $\beta < \alpha$, then $H(\eta \restriction \beta) \subseteq H(\eta \restriction \alpha)$.

Let us show that H is well define. Let $\eta, \xi \in \theta^\kappa$ and $\alpha \in C$ are such that $\eta \restriction \alpha = \xi \restriction \alpha$. Let $\alpha' = min(C \backslash (\alpha + 1))$. Clearly for all $\beta < \alpha'$ such that $\beta \notin C$, $F(\eta)(\beta) = 0 = F(\xi)(\beta)$. So $F(\eta) \restriction \alpha' \; (\beta) = 0 = F(\xi) \restriction \alpha' \; (\beta)$ for all $\beta \in \alpha' \backslash C$. On the other hand, by the definition of F, for all $\beta < \alpha'$ such that $\beta \in C$, $F(\eta)(\beta) = i$ and $F(\xi)(\beta) = j$, where $\eta \restriction \beta \in x_i^\beta$ and $\xi \restriction \beta \in x_j^\beta$. Since $\eta \restriction \beta = \xi \restriction \beta$ and E_β is an equivalence relation (since $\beta \in C$), $x_i^\beta = x_i^\beta$, and $i = j$. Thus $F(\eta) \restriction \alpha' \; (\beta) = F(\xi) \restriction \alpha' \; (\beta)$ for all $\beta \in \alpha' \cap C$. We conclude that $F(\eta) \restriction \alpha' \; = F(\xi) \restriction \alpha'$, $H(\eta \restriction \alpha) = H(\xi \restriction \alpha)$ and H is well defined.

Finally, from the way H was defined, for all $\alpha \in C$ and $\eta \in \theta^\kappa$, $F(\eta)(\beta) = H(\eta \restriction \alpha)(\beta)$ for all $\beta < min(S \backslash (\alpha + 1))$. ☐

Notice that for all $\beta < min(C)$ and $\eta \in \theta^\kappa$, $F(\eta)(\beta) = 0$. By Exercise 2.3, F is Lipschitz. ☐

3 Combinatorics

3.1 Filter reflection

Definition 3.1. *We say that a stationary set $S \subseteq \kappa$ reflects at α if $S \cap \alpha$ is stationary at α, where $cf(\alpha) > \omega$.*

We say that a stationary set $S \subseteq \kappa$ reflects to X if for all $\alpha \in X$, S reflects at α. We say that S strongly reflects to X if for all stationary $Z \subseteq S$ there is $Y \subseteq X$, such that Z reflects to Y.

Recall that the cofinality of an ordinal α, $cf(\alpha)$, is the smallest cardinal γ such that there is a function $G : \gamma \to \alpha$, such that for all $\beta < \alpha$, there is $\theta < \gamma$, such that $\beta < G(\theta)$. For all regular cardinal $\gamma < \kappa$, define S^κ_γ as the set of ordinals below κ with cofinality γ.

Lemma 3.2 (Aspero-Hyttinen-Weisnstein(Kulikov)-Moreno, [1] Proposition 2.8)**.** *Suppose $\gamma < \lambda < \kappa$ are regular cardinals If S^κ_γ strongly reflects to S^κ_λ, then $=^\kappa_\gamma \hookrightarrow_c =^\kappa_\lambda$.*

Proof. For all $\alpha \in S^\kappa_\lambda$, let E_α be the equivalence relation defined by

$$\eta E_\alpha \xi \iff \{\beta < \alpha \mid \eta(\beta) \neq \xi(\beta)\} \cap S^\kappa_\gamma \text{ is non-stationary in } \alpha.$$

Let $\langle x^\alpha_i \mid 0 < i < \kappa\rangle$ be an enumeration of the E_α-equivalence classes. Let us define the function $F : \kappa^\kappa \to \kappa^\kappa$ by

$$F(\eta)(\alpha) = \begin{cases} i & \text{if } \alpha \in S^\kappa_\lambda \text{ and } \eta \in x^\alpha_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that $\eta \ =^\kappa_\gamma\ \xi$ if and only if $F(\eta) \ =^\kappa_\lambda\ F(\xi)$.

Suppose $\eta \ =^\kappa_\gamma\ \xi$. There is a club $C \subseteq \kappa$, such that $C \cap S^\kappa_\gamma \subseteq \{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\}$. Thus for all $\alpha \in C \cap S^\kappa_\lambda$ limit in C, $C \cap S^\kappa_\gamma \cap \alpha \subseteq \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\}$ and $\eta E_\alpha \xi$. Therefore there is a club $D \subseteq \kappa$ (the limits of C) such that $D \cap S^\kappa_\lambda \subseteq \{\alpha < \kappa \mid F(\eta)(\alpha) = F(\xi)(\alpha)\}$. we conclude that $F(\eta) =^\kappa_\lambda F(\xi)$.

Suppose $\eta \ \neq^\kappa_\gamma\ \xi$. Then $Z = \{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cap S^\kappa_\gamma$ is stationary. By strong reflection, there is a stationary $Y \subseteq X$ such that Z reflects to Y. Thus, for all $\alpha \in Y$, $Z \cap \alpha$ is stationary in α. Since $Z \cap \alpha \subseteq \{\beta < \alpha \mid \eta(\beta) \neq \xi(\beta)\} \cap S^\kappa_\gamma$, for all $\alpha \in Y$, $\{\beta < \alpha \mid \eta(\beta) \neq \xi(\beta)\} \cap S^\kappa_\gamma$ is stationary in α. Therefore for all $\alpha \in Y$, η and ξ have different equivalence classes in E_α and $F(\eta)(\alpha) \neq F(\xi)(\alpha)$. We conclude that $F(\eta) \ \neq^\kappa_\lambda\ F(\xi)$.

Same as in Exercise 2.8, F is S^κ_λ-recursive and continuous. □

Definition 3.3. *$\mathcal{F} \subseteq \mathcal{P}(\delta)$ is a filter over δ if the following holds:*

- *$\delta \in \mathcal{F}$,*
- *for all $x \in \mathcal{F}$, if $x \subseteq y$, then $y \in \mathcal{F}$,*
- *if $x, y \in \mathcal{F}$, then $x \cap y \in \mathcal{F}$.*

Given a filter $\mathcal{F}$ over δ, we denote by $\mathcal{F}^+$ the set $\{A \subseteq \delta \mid \forall B \in \mathcal{F}(A \cap B \neq \emptyset)\}$.

Definition 3.4. *Let $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ be a filter over κ. For any set $\mathbf{R}$, $\mathcal{F}$ induces an equivalence relation over the space $\mathbf{R}^\kappa$. Let $\sim^{\mathbf{R}}_{\mathcal{F}}$ be the following relation:*

$$\eta \sim^{\mathbf{R}}_{\mathcal{F}} \xi \Leftrightarrow \exists W \in \mathcal{F} \ \ (W \subseteq \{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\})$$

Exercise 3.1. *Show that for any filter $\mathcal{F}$, $\sim^{\mathbf{R}}_{\mathcal{F}}$ is an equivalence relation.*

We say that an equivalence relation E is filtered if and only if there is a filter $\mathcal{F}$ such that $\eta\ E\ \xi \Leftrightarrow \eta \sim^{\mathbf{R}}_{\mathcal{F}} \xi$.

Exercise 3.2. *Show that the following are filtered equivalence relations:*

1. *id_2.*
2. *0_κ.*
3. *$E^{<\kappa}_0$.*
4. *$=^2_S$ where $S \subseteq \kappa$ is stationary.*

Exercise 3.3. *Show that 0_ϱ is not a filtered relation when $\varrho < \kappa$.*

Let us define $E^{<\kappa,\kappa}_0$, the *equivalence modulo bounded* over κ^κ, as:

$$E^{<\kappa,\kappa}_0 := \{(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \mid \exists \alpha < \kappa\ [\forall \beta > \alpha\ (\eta(\beta) = \xi(\beta)])\}.$$

Exercise 3.4. *1. Show that $E_0^{<\kappa,\kappa}$ is a filtered equivalence relation.*

2. Prove that for any stationary set $S \subseteq \kappa$, $E_0^{<\kappa,\kappa} \hookrightarrow_L =^\kappa_S$.

Definition 3.5. *Suppose $S \subseteq \kappa$ is a stationary set and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S\rangle$ is a sequence of filters, i.e. for all $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α. We say that $\vec{\mathcal{F}}$ captures clubs if and only if for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is non-stationary.*

Example 3.1. *Let $\omega < \lambda < \kappa$ be a regular cardinal. For all $\alpha \in S^\kappa_\lambda$, let $\mathcal{F}_\alpha$ be the club filter of α. Clearly $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S^\kappa_\lambda\rangle$ captures clubs.*

Definition 3.6. *Suppose $X, S \subseteq \kappa$ are stationary sets, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S\rangle$ is a sequence of filters. We say that X $\vec{\mathcal{F}}$-reflects to S if and only if $\vec{\mathcal{F}}$ captures clubs, and for every stationary set $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ is stationary.*

We say that X $\mathfrak{f}$-reflects to S if and only if there exists a sequence $\vec{\mathcal{F}}$ over a stationary subset $S' \subseteq S$ such that X $\vec{\mathcal{F}}$-reflects to S'.

Exercise 3.5. *Prove Lemma 3.7.*

Lemma 3.7 (Monotonicity, Fernandes-Moreno-Rinot, [3] Lemma 2.4)**.** *Suppose $Y \subseteq X \subseteq \kappa$ and $S \subseteq T \subseteq \kappa$ are stationary sets. If X $\mathfrak{f}$-reflects to S, then Y $\mathfrak{f}$-reflects to T.*

Lemma 3.8 (Fernandes-Moreno-Rinot, [3] Lemma 2.8)**.** *If X $\mathfrak{f}$-reflects to S, then $=^\kappa_X \hookrightarrow_L =^\kappa_S$.*

Proof. Suppose that $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S'\rangle$ witnesses that X $\mathfrak{f}$-reflects to S. For every $\alpha \in S'$, define an equivalence relation $\sim_\alpha$ over κ^α by letting $\eta \sim_\alpha \xi$ iff there is $W \in \mathcal{F}_\alpha$ such that $W \cap X \subseteq \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\}$. As there are at most $|\kappa^\alpha|$ many equivalence classes and as $\kappa^{<\kappa} = \kappa$, we can enumerate the equivalence classes $[\eta]_{\sim_\alpha}$, $\langle x^\alpha_i \mid 0 < i < \kappa\rangle$. Next, define a map $f : \kappa^\kappa \to \kappa^\kappa$ by letting for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$:

$$f(\eta)(\alpha) := \begin{cases} i & \text{if } \alpha \in S' \text{ and } [\eta \restriction \alpha]_{\sim_\alpha} = x^\alpha_i; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly f is Lipschitz and S'-recursive. To show that it is a reduction from $=^\kappa_X$ to $=^\kappa_S$, let η, ξ be arbitrary elements of κ^κ.

- $\eta =^\kappa_X \xi$**:** There is a club C such that $C \cap X \subseteq \{\beta < \kappa \mid \eta(\beta) = \xi(\beta)\}$. Since $\vec{\mathcal{F}}$ captures clubs, there is a club $D \subseteq \kappa$ such that, for all $\alpha \in D \cap S'$, $C \cap \alpha \in \mathcal{F}_\alpha$.

  Claim 3.9. *$D \cap \{\alpha \in S \mid f(\eta)(\alpha) \neq f(\xi)(\alpha)\} = \emptyset$, so $f(\eta) =^\kappa_S f(\xi)$.*

  Proof. Let $\alpha \in D$ be arbitrary. If $\alpha \notin S'$, then $f(\eta)(\alpha) = 0 = f(\xi)(\alpha)$.
  If $\alpha \in S'$, then for $W := C \cap \alpha$, we have that $W \in \mathcal{F}_\alpha$ and $W \cap X \subseteq \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\}$, so that $[\eta \restriction \alpha]_{\sim_\alpha} = [\xi \restriction \alpha]_{\sim_\alpha}$ and $f(\eta)(\alpha) = f(\xi)(\alpha)$. □

- $\eta \neq^\kappa_X \xi$**:** So $Y := \{\beta \in X \mid \eta(\beta) \neq \xi(\beta)\}$ is stationary. Since $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S'\rangle$ witnesses that X $\mathfrak{f}$-reflects to S, $T := \{\alpha \in S' \mid Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ is stationary. Now, for every $\alpha \in T$ and any $W \in \mathcal{F}_\alpha$, $W \cap Y \cap \alpha \neq \emptyset$. So that $W \cap Y \cap \alpha \subseteq W \cap X$, $W \cap X \nsubseteq \{\beta < \alpha \mid \eta(\beta) = \xi(\beta)\}$, and $[\eta \restriction \alpha]_{\sim_\alpha} \neq [\xi \restriction \alpha]_{\sim_\alpha}$. It follows that $T \subseteq \{\alpha \in S' \mid f(\eta)(\alpha) \neq f(\xi)(\alpha)\}$, so that $f(\eta) \neq_S f(\xi)$.

□

Exercise 3.6. *Prove Lemma 3.10.*

Lemma 3.10 (Fernandes-Moreno-Rinot, [3] Lemma 2.17)**.** *Suppose X, Y, Z are stationary subsets of κ, with $X \cap Y = \emptyset$. Prove the following:*

1. If X $\mathfrak{f}$-reflects to Y and Y $\mathfrak{f}$-reflects to X, then there is a function simultaneously witnessing

$$=_X \hookrightarrow_L =_Y \ \& \ =_Y \hookrightarrow_L =_X .$$

2. If Z $\mathfrak{f}$-reflects to Y and Z $\mathfrak{f}$-reflects to X, then there is a function simultaneously witnessing

$$=_Z \hookrightarrow_L =_Y \ \& \ =_Z \hookrightarrow_L =_X .$$

3.2 Diamond principle

Definition 3.11. *For a given cardinal λ and a stationary set $S \subseteq \lambda$, $\diamondsuit_\lambda(S)$ is the statement that there is a sequence $\langle D_\alpha \mid \alpha \in S\rangle$ such that*

- *For all $\alpha \in S$, $D_\alpha \subseteq \alpha$.*
- *For all $A \subseteq \lambda$, the set $\{\alpha \in S \mid D_\alpha = A \cap \alpha\}$ is stationary.*

Exercise 3.7. *Show that if λ is an infinite cardinal and $S \subseteq \lambda^+$ is a stationary set. Then $\diamondsuit_{\lambda^+}(S)$ implies $\lambda^+ = |\mathcal{P}(\lambda)| = 2^\lambda$.*

Lemma 3.12 (Friedman-Hyttinen-Weisnstein(Kulikov), [5] Theorem 60)**.** *Let $S \subseteq \kappa$ be stationary and suppose that $\diamondsuit_\kappa(S)$. Then*

$$E_0^{<\kappa} \hookrightarrow_L =^2_S$$

Proof. Let $\langle D_\alpha \mid \alpha \in S\rangle$ be a sequence that witnesses $\diamondsuit_\kappa(S)$. For all $\alpha \in S$, let $\eta_\alpha : \alpha \to 2$ be the function

$$\eta_\alpha(\beta) := \begin{cases} 1 & \text{if } \beta \in D_\alpha; \\ 0, & \text{otherwise.} \end{cases}$$

For all $\alpha \in S$ let $\mathcal{F}_\alpha$ be the filter $\{Z \subseteq \alpha \mid \exists \beta < \alpha\ (Z \cup \beta = \alpha)\}$, and $\sim_\alpha$ the equivalent relation induced by $\mathcal{F}_\alpha$. Define $f : 2^\kappa \to 2^\kappa$ by:

$$f(\eta)(\alpha) := \begin{cases} 1 & \text{if } \eta_\alpha \in [\eta \restriction \alpha]_{\sim_\alpha}; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that f is Lipschitz.

- Suppose $\eta\ E_0^{<\kappa}\ \xi$. Thus there is $\beta < \kappa$ such that for all $\alpha > \beta$, $\eta \restriction \alpha \ \sim_\alpha\ \xi \restriction \alpha$. Then, for all $\alpha > \beta$, $f(\eta)(\alpha) = f(\xi)(\alpha)$. In particular, for all $\alpha \in S \cap \beta$, so $f(\eta) =^2_S f(\xi)$.
- Suppose $\neg(\eta\ E_0^{<\kappa}\ \xi)$. There is an unbounded set $S \subseteq \kappa$, such that $\forall \alpha \in A$, $\eta(\alpha) \neq \xi(\alpha)$. So there is a club $C \subseteq \kappa$, such that $A \subseteq C$ and for all $\alpha \in C$, α a limit of C, $\neg(\eta \restriction \alpha \ \sim_\alpha\ \xi \restriction \alpha)$. Thus $[\eta \restriction \alpha]_{\sim_\alpha} \neq [\xi \restriction \alpha]_{\sim_\alpha}$. On the other hand, by $\diamondsuit_\kappa(S)$, the set

$$\begin{aligned} R &= \{\alpha < \kappa \mid \eta \restriction \alpha = \eta_\alpha\} \\ &= \{\alpha < \kappa \mid (\eta \restriction \alpha)^{-1}[1] = \eta_\alpha^{-1}[1]\} \\ &= \{\alpha < \kappa \mid \eta^{-1}[1] \cap \alpha = D_\alpha\} \end{aligned}$$

  is stationary. So, for all $\alpha \in C \cap R$, $\eta_\alpha \in [\eta \restriction \alpha]_{\sim_\alpha}$ and $\eta_\alpha \notin [\xi \restriction \alpha]_{\sim_\alpha}$. We conclude that for all $\alpha \in C \cap R$, $f(\eta)(\alpha) = 1$ and $f(\eta)(\alpha) = 0$. Since R is stationary, $C \cap R$ is stationary and $f(\eta) \neq^2_S f(\xi)$.

□

Definition 3.13. *We say that X $\vec{\mathcal{F}}$-*reflects with $\diamondsuit$ to S *iff $\vec{\mathcal{F}}$ captures clubs and there exists a sequence $\langle Y_\alpha \mid \alpha \in S\rangle$ such that, for every stationary $Y \subseteq X$, the set $\{\alpha \in S \mid Y_\alpha = Y \cap \alpha \ \&\ Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ is stationary.*

We say that X $\mathfrak{f}$-reflects with $\diamondsuit$ to S if and only if there exists a sequence $\vec{\mathcal{F}}$ over a stationary subset $S' \subseteq S$ such that X $\vec{\mathcal{F}}$-reflects with $\diamondsuit$ to S'.

Lemma 3.14 (Fernandes-Moreno-Rinot, [3] Claim 2.14.1)**.** *Let $X, S \subseteq \kappa$ be stationary sets such that X $\mathfrak{f}$-reflects with $\diamondsuit$ to S. There is $S' \subseteq S$ stationary, a sequence $\langle \eta_\alpha \mid \alpha \in S'\rangle$, and $\langle \bar{\mathcal{F}}_\alpha \mid \alpha \in S\rangle$ such that, for every stationary $Y \subseteq X$ and every $\eta \in \kappa^\kappa$, the set $\{\alpha \in S' \mid \eta_\alpha = \eta \restriction \alpha \ \&\ Y \cap \alpha \in \bar{\mathcal{F}}_\alpha\}$ is stationary.*

Proof. Let $S'' \subseteq \kappa$, $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S''\rangle$ and $\langle Y_\alpha \mid \alpha \in S''\rangle$ witness together that X $\mathfrak{f}$-reflects with $\diamondsuit$ to S. Let $S' := \{\alpha \in S'' \mid Y_\alpha \in \mathcal{F}^+_\alpha\}$. For each $\alpha \in S'$, let $\bar{\mathcal{F}}_\alpha$ be the filter over α generated by $\mathcal{F}_\alpha \cup \{Y_\alpha\}$.

Let C be the set of limit points of X and $B := X \backslash C$, so, C is a club and B is not stationary and has cardinality κ. Let $\{a_\beta \mid \beta \in B\}$ be an enumeration of $\kappa^{<\kappa}$. Then, for each $\alpha \in S'$, let $\eta_\alpha := (\bigcup\{a_\beta \mid \beta \in Y_\alpha \cap B\}) \cap (\alpha \times \alpha)$.

Claim 3.15. *$\langle \eta_\alpha \mid \alpha \in S'\rangle$ is as wanted.*

Proof. Let $\eta \in \kappa^\kappa$ and $Y \subseteq X$ stationary. Let $f : \kappa \to B$ be the unique function to satisfy that, for all $\epsilon < \kappa$, $a_{f(\epsilon)} = \eta \restriction \epsilon$. Notice that $Y \cap C$ is a stationary subset of X disjoint from $\mathrm{Im}(f)$. In particular, $Y' = (Y \cap C) \cup \mathrm{Im}(f)$ is a stationary subset of X, and hence $G := \{\alpha \in S' \mid Y_\alpha = Y' \cap \alpha \ \&\ Y' \cap \alpha \in \mathcal{F}^+_\alpha\}$ is a stationary subset of S'.

Now, as $\vec{\mathcal{F}}$ captures clubs, let us fix a club $D \subseteq \kappa$ such that, for all $\alpha \in D \cap S'$, $C \cap \alpha \in \mathcal{F}_\alpha$. Therefore $T = \{\alpha \in G \cap D \mid f[\alpha] \subseteq \alpha \ \&\ \eta[\alpha] \subseteq \alpha\}$ is a stationary subset of S'. Let us show that for all $\alpha \in T$, $\eta_\alpha = \eta \restriction \alpha$ and $Y \cap \alpha \in \bar{\mathcal{F}}_\alpha$. Let $\alpha \in T$.

- Since $\alpha \in D$, $C \cap \alpha \in \mathcal{F}_\alpha \subseteq \bar{\mathcal{F}}_\alpha$. Since $\alpha \in G$, $Y' \cap \alpha = Y_\alpha \in \bar{\mathcal{F}}_\alpha$. Therefore, the intersection $Y' \cap C \cap \alpha$ is in $\bar{\mathcal{F}}_\alpha$. But $Y' \cap C \cap \alpha = Y \cap C \cap \alpha$, and hence the superset $Y \cap \alpha$ is in $\bar{\mathcal{F}}_\alpha$, as well.
- Since $\alpha \in G$, $Y_\alpha = Y' \cap \alpha$ and $Y_\alpha \cap B = \mathrm{Im}(f) \cap \alpha$. Since $f[\alpha] \subseteq \alpha$, $f[\alpha] \subseteq Y_\alpha \cap B \subseteq \mathrm{Im}(f)$. As $\eta[\alpha] \subseteq \alpha$, we get that $\eta \restriction \alpha = \eta \cap (\alpha \times \alpha)$. Recalling the definition of f and the definition of η_α, it follows that $\eta \restriction \alpha \subseteq \eta_\alpha \subseteq \eta$, so that $\eta_\alpha = \eta \restriction \alpha$.

□

□

Exercise 3.8. *Prove Lemma 3.16.*

Theorem 3.16 (Fernandes-Moreno-Rinot, [3] Theorem 2.14)**.** *If X $\mathfrak{f}$-reflects with $\diamondsuit$ to S, then $=^\kappa_X \hookrightarrow_L =^2_S$. Hint: Similar to Lemma 3.12). Use the previous lemma to guess the equivalence classes.*

Exercise 3.9. *Suppose $\diamondsuit_\kappa(S)$ holds. Show that the following holds: there is a sequence $\langle f_\alpha \mid \alpha \in S\rangle$ such that*

- *for all $\alpha \in S$, $f_\alpha : \alpha \to \alpha$,*
- *for all $f \in \kappa^\kappa$, the set $\{\alpha \in S \mid f_\alpha = f \restriction \alpha\}$ is stationary.*

Exercise 3.10. *Let id_κ be the identity relation in the space κ^κ. Show that $id_\kappa \hookrightarrow_L id_2$.*

3.3 Reflection of Π^1_2-sentences

In this session we will focus on proving the consistency of κ-Borel$^* = \Sigma^1_1(\kappa)$. This was initially proved by Friedman-Hyttinen-Weisnstein in [5].

Theorem 3.17 (Friedman-Hyttinen-Weisnstein(Kulikov), [5] Theorem 18)**.** *If $V = L$, then κ-Borel$^* = \Sigma^1_1(\kappa)$.*

We will show another proof which shows that κ-Borel$^* = \Sigma^1_1(\kappa)$ holds under certain reflection principle.

A Π^1_2-sentence ϕ is a formula of the form $\forall X \exists Y \varphi$ where φ is a first-order sentence over a relational language $\mathcal{L}$ as follows:

- $\mathcal{L}$ has a predicate symbol ϵ of arity 2;
- $\mathcal{L}$ has a predicate symbol $\mathbb{X}$ of arity $m(\mathbb{X})$;
- $\mathcal{L}$ has a predicate symbol $\mathbb{Y}$ of arity $m(\mathbb{Y})$;
- $\mathcal{L}$ has infinitely many predicate symbols $(\mathbb{A}_n)_{n\in\omega}$, each $\mathbb{A}_n$ is of arity $m(\mathbb{A}_n)$.

Definition 3.18. *A cardinal λ is Π^1_2-indescribable if for every Π^1_2-sentence ϕ and a set $A \subseteq V_\lambda$ with $(V_\kappa, \in, A) \models \phi$, there is $\alpha < \kappa$ such that $(V_\alpha, \in, A \cap \alpha) \models \phi$.*

Exercise 3.11. *Show that if κ is Π^1_2-indescernible cardinal, then $Reg(\kappa) = \{\alpha < \kappa \mid cf(\alpha) = \alpha\}$, the set of regular cardinals below κ, is stationary.*

We say that an equivalence relation E is Σ^1_1-complete if it is a Σ^1_1 equivalence relation and for all Σ^1_1 equivalence relation, R, $R \hookrightarrow_B E$.

Let us show that if κ is Π^1_2-indescernible cardinal, then $=^\kappa_{Reg}$ is a Σ^1_1-complete equivalence relation.

Theorem 3.19 (Aspero-Hyttinen-Weisnstein(Kulikov)-Moreno, [1] Thm 3.7)**.** *If κ is a Π^1_2–indescribable cardinal, then $=^\kappa_{Reg}$ is $\Sigma^1_1(\kappa)$–complete.*

Proof. Let E be a $\Sigma^1_1(\kappa)$ equivalence relation on κ^κ. Then there is a closed set C on $\kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa$ such that $\eta\ E\ \xi$ if and only if there exists $\zeta \in \kappa^\kappa$ such that $(\eta, \xi, \zeta) \in C$. Let us define $U = \{(\eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha) \mid (\eta, \xi, \zeta) \in C \ \&\ \alpha < \kappa\}$, and for every $\gamma < \kappa$ define $C_\gamma = \{(\eta, \xi, \zeta) \in \gamma^\gamma \times \gamma^\gamma \times \gamma^\gamma \mid \forall \alpha < \gamma\ (\eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha) \in U\}$. Let $E_\gamma \subset \gamma^\gamma \times \gamma^\gamma$ be the relation defined by $(\eta, \xi) \in E_\gamma$ if and only if there exists $\zeta \in \gamma^\gamma$ such that $(\eta, \xi, \zeta) \in C_\gamma$. Since E is an equivalence relation, it follows that E_γ is reflexive and symmetric, but not necessary transitive. Let $\langle x^\alpha_i \mid 0 < i < \kappa\rangle$ be an enumeration fo the equivalence classes of E_α, when E_α is an equivalence relation. Let us define the reduction by

$$F(\eta)(\alpha) = \begin{cases} i \text{ if } E_\alpha \text{ is an equivalence relation,} \eta \restriction \alpha \in \alpha^\alpha \text{ and } \eta \in x^\alpha_i \\ 0 \text{ otherwise.} \end{cases}$$

Let us prove that if $(\eta, \xi) \in E$, then $F(\eta)\ =^\kappa_{reg}\ F(\xi)$. Suppose $(\eta, \xi) \in E$. Then there is $\zeta \in \kappa^\kappa$ such that $(\eta, \xi, \zeta) \in C$ and for all $\alpha < \kappa$ we have that $(\eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha) \in U$. On the other hand, we know that there is a club D such that for all $\alpha \in D \cap Reg(\kappa)$, $\eta \restriction \alpha$, $\xi \restriction \alpha$, $\zeta \restriction \alpha \in \alpha^\alpha$. We conclude that for all $\alpha \in D \cap Reg(\kappa)$, if E_α is an equivalence relation, then $(\eta, \xi) \in E_\alpha$. Therefore, for all $\alpha \in D \cap Reg(\kappa)$, $F(\eta)(\alpha) = F(\xi)(\alpha)$, so $F(\eta) \ =^\kappa_{Reg} \ F(\xi)$. Let us prove that if $(\eta, \xi) \notin E$, then $F(\eta) \ \neq^\kappa_{Reg} \ F(\xi)$. Suppose η, $\xi \in \kappa^\kappa$ are such that $(\eta, \xi) \notin E$. We know that there is a club D such that for all $\alpha \in D \cap Reg(\kappa)$, $\eta \restriction \alpha$, $\xi \restriction \alpha \in \alpha^\alpha$.

Notice that because C is closed $(\eta, \xi) \notin E$ is equivalent to

$$\forall \zeta \in \kappa^\kappa \ (\exists \alpha < \kappa \ (\eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha) \notin U),$$

so the sentence $(\eta, \xi) \notin E$ is a Π^1_1 property of the structure $(V_\kappa, \in, U, \eta, \xi)$. On the other hand, the sentence $\forall \zeta_1, \zeta_2, \zeta_3 \in \kappa^\kappa[((\zeta_1, \zeta_2) \in E \wedge (\zeta_2, \zeta_3) \in E) \rightarrow (\zeta_1, \zeta_3) \in E]$ is equivalent to the sentence $\forall \zeta_1, \zeta_2, \zeta_3, \theta_1, \theta_2 \in \kappa^\kappa[\exists \theta_3 \in \kappa^\kappa(\psi_1 \vee \psi_2 \vee \psi_3)]$, where ψ_1, ψ_2 and ψ_3 are, respectively, the formulas $\exists \alpha_1 < \kappa \ (\zeta_1 \restriction \alpha_1, \zeta_2 \restriction \alpha_1, \theta_1 \restriction \alpha_1) \notin U$, $\exists \alpha_2 < \kappa \ (\zeta_2 \restriction \alpha_2, \zeta_3 \restriction \alpha_2, \theta_2 \restriction \alpha_2) \notin U$, and $\forall \alpha_3 < \kappa \ (\zeta_1 \restriction \alpha_3, \zeta_3 \restriction \alpha_3, \theta_3 \restriction \alpha_3) \in U$. Therefore, the sentence $\forall \zeta_1, \zeta_2, \zeta_3 \in \kappa^\kappa[((\zeta_1, \zeta_2) \in E \wedge (\zeta_2, \zeta_3) \in E) \rightarrow (\zeta_1, \zeta_3) \in E]$ is a Π^1_2 property of the structure $(V_\kappa, \in, U)$. It follows that the sentence

$$(D \text{ is unbounded in } \kappa) \wedge ((\eta, \xi) \notin E) \wedge (E \text{ is an equivalence relation}) \wedge (\kappa \text{ is regular})$$

is a Π^1_2 property of the structure $(V_\kappa, \in, U, \eta, \xi)$. By Π^1_2 reflection, we know that there are stationary many $\gamma \in Reg(\kappa)$ such that γ is a limit point of D, E_γ is an equivalence relation, and $(\eta \restriction \gamma, \xi \restriction \gamma) \notin E_\gamma$. We conclude that there are stationary many $\gamma \in Reg(\kappa)$ such that $f_\gamma(\eta) \neq f_\gamma(\xi)$, and hence $F(\eta) \ \neq^\kappa_{reg} \ F(\eta)$ ☐

As we can see from the previous theorem, Π^1_2 reflection implies that $=^\kappa_{Reg}$ is $\Sigma^1_1(\kappa)$-complete. Unfortunately $=^\kappa_{Reg}$ is not necessarily κ-Borel*. As we saw, $=^\kappa_\omega$ is a κ-Borel* equivalence relation. Therefore, if there is a Π^1_2 reflection notion on the set $\{\alpha < \kappa \mid cf(\alpha) = \omega\}$, then we conclude that κ-Borel* $= \Sigma^1_1(\kappa)$. Let us define a notion of reflection on ordinals of cofinality ω.

Definition 3.20. *For sets N and x, we say that N* sees *x iff N is transitive, p.r.-closed, and $x \cup \{x\} \subseteq N$.*

Suppose that a set N sees an ordinal α, and that $\phi = \forall X \exists Y \varphi$ is a Π^1_2-sentence, where φ is a first-order sentence in the above-mentioned language $\mathcal{L}$. For every sequence $(A_n)_{n\in\omega}$ such that, for all $n \in \omega$, $A_n \subseteq \alpha^{m(\mathbb{A}_n)}$, we write

$$\langle \alpha, \in, (A_n)_{n\in\omega} \rangle \models_N \phi$$

to express that the two hold:

1. $(A_n)_{n\in\omega} \in N$;
2. $\langle N, \in \rangle \models (\forall X \subseteq \alpha^{m(\mathbb{X})})(\exists Y \subseteq \alpha^{m(\mathbb{Y})})[\langle \alpha, \in, X, Y, (A_n)_{n\in\omega} \rangle \models \varphi]$, where:
   - $\in$ is the interpretation of ϵ;
   - X is the interpretation of $\mathbb{X}$;
   - Y is the interpretation of $\mathbb{Y}$, and
   - for all $n \in \omega$, A_n is the interpretation of $\mathbb{A}_n$.

We write α^+ for $|\alpha|^+$, and write $\langle \alpha, \in, (A_n)_{n\in\omega} \rangle \models \phi$ for

$$\langle \alpha, \in, (A_n)_{n\in\omega} \rangle \models_{H_{\alpha^+}} \phi.$$

Definition 3.21. *Let κ be a regular and uncountable cardinal, and $S \subseteq \kappa$ stationary.*
$\mathrm{Dl}^*_S(\Pi^1_2)$ asserts the existence of a sequence $\vec{N} = \langle N_\alpha \mid \alpha \in S \rangle$ satisfying the following:

1. *for every $\alpha \in S$, N_α is a set of cardinality $< \kappa$ that sees α;*
2. *for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$;*
3. *whenever $\langle \kappa, \in, (A_n)_{n\in\omega} \rangle \models \phi$, with ϕ a Π^1_2-sentence, there are stationarily many $\alpha \in S$ such that $|N_\alpha| = |\alpha|$ and $\langle \alpha, \in, (A_n \cap (\alpha^{m(\mathbb{A}_n)}))_{n\in\omega} \rangle \models_{N_\alpha} \phi$.*

The principle $\mathrm{Dl}^*_S(\Pi^1_2)$ provide us the reflection principle that we need, let us show that there is a Σ^1_1-complete quasi-order of 2^κ. If Q_1 and Q_2 are quasi-orders on $\mathbb{B}_1, \mathbb{B}_2 \in \{2^\kappa, \kappa^\kappa\}$, respectively, then we say that Q_1 is *Borel-reducible* to Q_2 if there exists a κ-Borel map $f \colon \mathbb{B}_1 \to \mathbb{B}_1$ such that for all $\eta, \xi \in 2^\kappa$ we have $\eta Q_1 \xi \iff f(\eta) Q_2 f(\xi)$ and this is also denoted by $Q_1 \hookrightarrow_B Q_2$.

Definition 3.22. *Given a stationary subset $S \subseteq \kappa$, we define a quasi-order $\subseteq^S$ over 2^κ by letting, for any two elements $\eta : \kappa \to 2$ and $\xi : \kappa \to 2$,*

$$\eta \subseteq^S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \text{ is nonstationary.}$$

Lemma 3.23 (Transversal lemma, Fernandes-Moreno-Rinot, [2], Prop 3.1). *Suppose that $\langle N_\alpha \mid \alpha \in S\rangle$ is a $\mathrm{Dl}^*_S(\Pi^1_2)$-sequence, for a given stationary $S \subseteq \kappa$. For every Π^1_2-sentence ϕ, there exists a transversal $\langle \eta_\alpha \mid \alpha \in S\rangle \in \prod_{\alpha\in S} N_\alpha$ satisfying the following.*

For every $\eta \in \kappa^\kappa$, whenever $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle \models \phi$, there are stationarily many $\alpha \in S$ such that

1. *$\eta_\alpha = \eta \restriction \alpha$, and*
2. *$\langle \alpha, \in, (A_n \cap (\alpha^{m(\mathbb{A}_n)}))_{n\in\omega}\rangle \models_{N_\alpha} \phi$.*

Exercise 3.12. *There is a first-order sentence ψ_{fnc} in the language with binary predicate symbols ϵ and $\mathbb{X}$ such that, for every ordinal α and every $X \subseteq \alpha \times \alpha$,*

$$(X \textit{ is a function from } \alpha \textit{ to } \alpha) \textit{ iff } (\langle \alpha, \in, X\rangle \models \psi_{\text{fnc}}).$$

Exercise 3.13. *Let α be an ordinal. Suppose that ϕ is a Σ^1_1-sentence involving a predicate symbol $\mathbb{A}$ and two binary predicate symbols $\mathbb{X}_0, \mathbb{X}_1$. Denote $R_\phi := \{(X_0, X_1) \mid \langle \alpha, \in, A, X_0, X_1\rangle \models \phi\}$. Then there are Π^1_2-sentences $\psi_{\text{Reflexive}}$ and $\psi_{\text{Transitive}}$ such that:*

1. *$(R_\phi \supseteq \{(\eta,\eta) \mid \eta \in \alpha^\alpha\})$ iff $(\langle \alpha, \in, A\rangle \models \psi_{\text{Reflexive}})$;*
2. *$(R_\phi$ is transitive$)$ iff $(\langle \alpha, \in, A\rangle \models \psi_{\text{Transitive}})$.*

Definition 3.24. *Denote by $\mathrm{Lev}_3(\kappa)$ the set of level sequences in $\kappa^{<\kappa}$ of length 3:*

$$\mathrm{Lev}_3(\kappa) := \bigcup_{\tau<\kappa} \kappa^\tau \times \kappa^\tau \times \kappa^\tau.$$

Fix an injective enumeration $\{\ell_\delta \mid \delta < \kappa\}$ of $\mathrm{Lev}_3(\kappa)$. For each $\delta < \kappa$, we denote $\ell_\delta = (\ell^0_\delta, \ell^1_\delta, \ell^2_\delta)$. We then encode each $T \subseteq \mathrm{Lev}_3(\kappa)$ as a subset of κ^5 via:

$$T_\ell := \{(\delta, \beta, \ell^0_\delta(\beta), \ell^1_\delta(\beta), \ell^2_\delta(\beta)) \mid \delta < \kappa, \ell_\delta \in T, \beta \in \mathrm{dom}(\ell^0_\delta)\}.$$

Theorem 3.25 (Fernandes-Moreno-Rinot, [2], Thm 3.5). *Suppose $\mathrm{Dl}^*_S(\Pi^1_2)$ holds for a given stationary $S \subseteq \kappa$. For every analytic quasi-order Q over κ^κ, $Q \hookrightarrow_B \subseteq^S$.*

Proof. Let Q be an analytic quasi-order over κ^κ. Fix a tree T on $\kappa^{<\kappa} \times \kappa^{<\kappa} \times \kappa^{<\kappa}$ such that $Q = \mathrm{pr}([T])$, that is,

$$(\eta, \xi) \in Q \iff \exists \zeta \in \kappa^\kappa\ \forall \tau < \kappa\ (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau) \in T.$$

We shall be working with a first-order language having a 5-ary predicate symbol $\mathbb{A}$ and binary predicate symbols $\mathbb{X}_0, \mathbb{X}_1, \mathbb{X}_2$ and ϵ. By Exercise 3.12, for each $i < 3$, let us fix a sentence ψ^i_{fnc} concerning the binary predicate symbol $\mathbb{X}_i$ instead of $\mathbb{X}$, so that

$$(X_i \in \kappa^\kappa) \text{ iff } (\langle \kappa, \in, A, X_0, X_1, X_2\rangle \models \psi^i_{\text{fnc}}).$$

Define a sentence φ_Q to be the conjunction of four sentences: ψ^0_{fnc}, ψ^1_{fnc}, ψ^2_{fnc}, and

$$\forall\tau\exists\delta\forall\beta[\epsilon(\beta,\tau) \to \exists\gamma_0\exists\gamma_1\exists\gamma_2(\mathbb{X}_0(\beta,\gamma_0) \wedge \mathbb{X}_1(\beta,\gamma_1) \wedge \mathbb{X}_2(\beta,\gamma_2) \wedge \mathbb{A}(\delta,\beta,\gamma_0,\gamma_1,\gamma_2))].$$

Set $A := T_\ell$ as in Definition 3.24. Evidently, for all $\eta, \xi, \zeta \in \mathcal{P}(\kappa\times\kappa)$, we get that

$$\langle \kappa, \in, A, \eta, \xi, \zeta\rangle \models \varphi_Q$$

iff the two hold:

1. $\eta, \xi, \zeta \in \kappa^\kappa$, and
2. for every $\tau < \kappa$, there exists $\delta < \kappa$, such that $\ell_\delta = (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau)$ is in T.

Let $\phi_Q := \exists X_2(\varphi_Q)$. Then ϕ_Q is a Σ^1_1-sentence involving predicate symbols $\mathbb{A}, \mathbb{X}_0, \mathbb{X}_1$ and ϵ for which the induced binary relation

$$R_{\phi_Q} := \{(\eta, \xi) \in (\mathcal{P}(\kappa\times\kappa))^2 \mid \langle \kappa, \in, A, \eta, \xi\rangle \models \phi_Q\}$$

coincides with the quasi-order Q. Now, appeal to Exercise 3.13 with ϕ_Q to receive the corresponding Π^1_2-sentences $\psi_{\text{Reflexive}}$ and $\psi_{\text{Transitive}}$. Then, consider the following two Π^1_2-sentences:

- $\psi^0_Q := \psi_{\text{Reflexive}} \wedge \psi_{\text{Transitive}} \wedge \phi_Q$, and
- $\psi^1_Q := \psi_{\text{Reflexive}} \wedge \psi_{\text{Transitive}} \wedge \neg(\phi_Q)$.

Let $\vec{N} = \langle N_\alpha \mid \alpha \in S\rangle$ be a $\mathrm{Dl}^*_S(\Pi^1_2)$-sequence. Appeal to Lemma 3.23 with the Π^1_2-sentence ψ^1_Q to obtain a corresponding transversal $\langle \eta_\alpha \mid \alpha \in S\rangle \in \prod_{\alpha\in S} N_\alpha$. Note that we may assume that, for all $\alpha \in S$, $\eta_\alpha \in {}^\alpha\alpha$, as this does not harm the key feature of the chosen transversal.

For each $\eta \in \kappa^\kappa$, let

$$Z_\eta := \{\alpha \in S \mid A \cap \alpha^5 \text{ and } \eta \restriction \alpha \text{ are in } N_\alpha\}.$$

Claim 3.26. *Suppose $\eta \in \kappa^\kappa$. Then $S \setminus Z_\eta$ is nonstationary.*

Proof. Fix primitive-recursive bijections $c : \kappa^2 \leftrightarrow \kappa$ and $d : \kappa^5 \leftrightarrow \kappa$. Given $\eta \in \kappa^\kappa$, consider the club D_0 of all $\alpha < \kappa$ such that:

- $\eta[\alpha] \subseteq \alpha$;
- $c[\alpha \times \alpha] = \alpha$;
- $d[\alpha \times \alpha \times \alpha \times \alpha \times \alpha] = \alpha$.

Now, as $c[\eta]$ is a subset of κ, by the choice $\vec{N}$, we may find a club $D_1 \subseteq \kappa$ such that, for all $\alpha \in D_1 \cap S$, $c[\eta] \cap \alpha \in N_\alpha$. Likewise, we may find a club $D_2 \subseteq \kappa$ such that, for all $\alpha \in D_2 \cap S$, $d[A] \cap \alpha \in N_\alpha$.

For all $\alpha \in S \cap D_0 \cap D_1 \cap D_2$, we have

- $c[\eta \restriction \alpha] = c[\eta \cap (\alpha \times \alpha)] = c[\eta] \cap c[\alpha \times \alpha] = c[\eta] \cap \alpha \in N_\alpha$, and
- $d[A \cap \alpha^5] = d[A] \cap d[\alpha^5] = d[A] \cap \alpha \in N_\alpha$.

As N_α is p.r.-closed, it then follows that $\eta \restriction \alpha$ and $A \cap \alpha^5$ are in N_α. Thus, we have shown that $S \setminus Z_\eta$ is disjoint from the club $D_0 \cap D_1 \cap D_2$. □

For all $\eta \in \kappa^\kappa$ and $\alpha \in Z_\eta$, let:

$$\mathcal{P}_{\eta,\alpha} := \{p \in \alpha^\alpha \cap N_\alpha \mid \langle \alpha, \in, A \cap \alpha^5, p, \eta \restriction \alpha\rangle \models_{N_\alpha} \psi^0_Q\}.$$

Finally, define a function $f : \kappa^\kappa \to 2^\kappa$ by letting, for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$,

$$f(\eta)(\alpha) := \begin{cases} 1, & \text{if } \alpha \in Z_\eta \text{ and } \eta_\alpha \in \mathcal{P}_{\eta,\alpha}; \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 3.14. *f is Borel.*

Claim 3.27. *Suppose $(\eta, \xi) \in Q$. Then $f(\eta) \subseteq^S f(\xi)$.*

Proof. As $(\eta, \xi) \in Q$, let us fix $\zeta \in \kappa^\kappa$ such that, for all $\tau < \kappa$, $(\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau) \in T$. Define a function $g : \kappa \to \kappa$ by letting, for all $\tau < \kappa$,

$$g(\tau) := \min\{\delta < \kappa \mid \ell_\delta = (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau)\}.$$

As $(S \setminus Z_\eta)$, $(S \setminus Z_\xi)$ and $(S \setminus Z_\zeta)$ are nonstationary, let us fix a club $C \subseteq \kappa$ such that $C \cap S \subseteq Z_\eta \cap Z_\xi \cap Z_\zeta$. Consider the club $D := \{\alpha \in C \mid g[\alpha] \subseteq \alpha\}$. We shall show that, for every $\alpha \in D \cap S$, if $f(\eta)(\alpha) = 1$ then $f(\xi)(\alpha) = 1$.

Fix an arbitrary $\alpha \in D \cap S$ satisfying $f(\eta)(\alpha) = 1$. In effect, the following three conditions are satisfied:

1. $\langle \alpha, \in, A \cap \alpha^5\rangle \models_{N_\alpha} \psi_{\mathrm{Reflexive}}$,
2. $\langle \alpha, \in, A \cap \alpha^5\rangle \models_{N_\alpha} \psi_{\mathrm{Transitive}}$, and
3. $\langle \alpha, \in, A \cap \alpha^5, \eta_\alpha, \eta \restriction \alpha\rangle \models_{N_\alpha} \phi_Q$.

In addition, since α is a closure point of g, by definition of φ_Q, we have

$$\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha\rangle \models \varphi_Q.$$

As $\alpha \in S$ and φ_Q is first-order,

$$\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha\rangle \models_{N_\alpha} \varphi_Q,$$

so that, by definition of ϕ_Q,

$$\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha\rangle \models_{N_\alpha} \phi_Q.$$

By combining the preceding with clauses (2) and (3) above, we infer that the following holds, as well:

(4) $\langle \alpha, \in, A \cap \alpha^5, \eta_\alpha, \xi \restriction \alpha \rangle \models_{N_\alpha} \phi_Q$.

Altogether, $f(\xi)(\alpha) = 1$, as sought. □

Claim 3.28. *Suppose $(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \setminus Q$. Then $f(\eta) \not\subseteq^S f(\xi)$.*

Proof. As $(S \setminus Z_\eta)$ and $(S \setminus Z_\xi)$ are nonstationary, let us fix a club $C \subseteq \kappa$ such that $C \cap S \subseteq Z_\eta \cap Z_\xi$. As Q is a quasi-order and $(\eta, \xi) \notin Q$, we have:

1. $\langle \kappa, \in, A \rangle \models \psi_{\text{Reflexive}}$,
2. $\langle \kappa, \in, A \rangle \models \psi_{\text{Transitive}}$, and
3. $\langle \kappa, \in, A, \eta, \xi \rangle \models \neg(\phi_Q)$.

so that, altogether,

$$\langle \kappa, \in, A, \eta, \xi \rangle \models \psi^1_Q.$$

Then, by the choice of the transversal $\langle \eta_\alpha \mid \alpha \in S \rangle$, there is a stationary subset $S' \subseteq S \cap C$ such that, for all $\alpha \in S'$:

1. $\langle \alpha, \in, A \cap \alpha^5 \rangle \models_{N_\alpha} \psi_{\text{Reflexive}}$,
2. $\langle \alpha, \in, A \cap \alpha^5 \rangle \models_{N_\alpha} \psi_{\text{Transitive}}$,
3. $\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha \rangle \models_{N_\alpha} \neg(\phi_Q)$, and
4. $\eta_\alpha = \eta \restriction \alpha$.

By Clauses (3') and (4'), we have that $\eta_\alpha \notin \mathcal{P}_{\xi,\alpha}$, so that $f(\xi)(\alpha) = 0$.
By Clauses (1'), (2') and (4'), we have that $\eta_\alpha \in \mathcal{P}_{\eta,\alpha}$, so that $f(\eta)(\alpha) = 1$.
Altogether, $\{\alpha \in S \mid f(\eta)(\alpha) > f(\xi)(\alpha)\}$ covers the stationary set S', so that $f(\eta) \not\subseteq^S f(\xi)$. □

This completes the proof of Theorem 3.25 □

Corollary 3.29. *Suppose $\text{Dl}^*_S(\Pi^1_2)$ holds for a given stationary $S \subseteq \kappa$.*
For every analytic equivalence relation E over κ^κ, $E \hookrightarrow_B =^2_S$.

As we have seen, the equivalence relations $=^\kappa_\mu$ and $=^2_\mu$ play a crucial role. It is clear that $\text{Dl}^*_\mu(\Pi^1_2)$ implies $=^\kappa_\mu \hookrightarrow_B =^2_\mu$.

Question 3.30. *Is $=^\kappa_\mu \hookrightarrow_B =^2_\mu$ a theorem of ZFC?*

4 The Isomorphism relation

Denote by $S^m(A)$ the set of all consistent types over A in m variables (modulo change of variables), and $S(A) = \cup_{m<\omega} S^m(A)$.

- We say that T is ξ-stable if for any set A, $|A| \leq \xi$, $|S(A)| \leq \xi$.
- We say that T is stable if there is an infinite ξ, such that T is ξ-stable.
- We say that T is unstable if there is no infinite ξ, such that T is ξ-stable.
- We say that T is superstable is there is an infinite ξ such that for all $\xi' > \xi$, T is ξ'-stable.

Definition 4.1 (OTOP)**.** *A theory T has the omitting type order property (OTOP) if there is a sequence $(\varphi_m)_{m<\omega}$ of first order formulas such that for every linear order l there is a model $\mathcal{M}$ and n-tuples a_t $(t \in l)$ of members of $\mathcal{M}$, $n < \omega$, such that $s < t$ if and only if there is a k-tuple c of members of $\mathcal{M}$, $k < \omega$, such that for every $m < \omega$,*

$$\mathcal{M} \models \varphi_m(c, a_s, a_t).$$

The non-forking notion $\downarrow$ and the isolation notion F^a_ω (Chapter 4 [19]) are needed to define the DOP.

Definition 4.2 (DOP)**.** *A theory T has the dimensional order property (DOP) if there are F^a_ω-saturated models $(M_i)_{i<3}$, $M_0 \subseteq M_1 \cap M_2$, $M_1 \downarrow_{M_0} M_2$, and the F^a_ω-prime model over $M_1 \cup M_2$ is not F^a_ω-minimal over $M_1 \cup M_2$.*

Definition 4.3.

- *We say that T is classifiable if T is superstable without DOP and without OTOP. These theories are diveded into:*
  - *shallow;*
  - *non-shallow (deep).*
- *We say that T is non-classifiable if it satisfies one of the following:*
  1. *T is stable unsuperstable;*
  2. *T is superstable and has DOP;*
  3. *T is superstable and has OTOP;*
  4. *T is unstable.*

Theorem 4.4 (Main Gap, Shelah [19, XII, Theorem 6.1]). *Let T be a first order countable complete theory and denote by $I(\lambda, T)$ the number of non-isomorphic models of T of size λ.*

1. *If T is not superstable or (is superstable) deep or has the DOP or has the OTOP, then for every uncountable λ, $I(\lambda, T) = 2^\lambda$.*
2. *If T is shallow superstable without the DOP and without the OTOP (i.e. classifiable and shallow), then for every $\alpha > 0$, $I(\aleph_\alpha, T) < \beth_{\omega_1}(|\alpha|)$.*

Theorem 4.5 (Morley's Conjecture, Shelah [19, XIII, Theorem 3.7]). *Let T be a countable complete first-order theory. Then for $\lambda > \mu \geq \aleph_0$, $I(\lambda, T) \geq I(\mu, T)$ except when $\lambda > \mu = \aleph_0$, T is complete, $\aleph_1$-categorical not $\aleph_0$-categorical.*

4.1 Coding structures

We can code structures of any size (not bigger than κ) with elements of κ^κ.

Definition 4.6. *Let $\omega \leq \mu \leq \kappa$ be a cardinal and $\mathbb{L} = \{Q_m \mid m \in \omega\}$ be a countable relational language. Fix a bijection π_μ between $\mu^{<\omega}$ and μ. For every $\eta \in \kappa^\kappa$ define the structure $\mathcal{A}_{\eta\restriction\mu}$ with domain μ as follows: For every tuple $(a_1, a_2, \dots, a_n)$ in μ^n*

$$(a_1, a_2, \dots, a_n) \in Q_m^{\mathcal{A}_{\eta\restriction\mu}} \Leftrightarrow Q_m \text{ has arity } n \text{ and } \eta(\pi_\mu(m, a_1, a_2, \dots, a_n)) > 0.$$

Notice that the structure $\mathcal{A}_\eta \restriction \alpha$ is not necessary coded by the function $\eta \restriction \alpha$.

Exercise 4.1. *There is a club C_π such that for all $\alpha \in C_\pi$, $\mathcal{A}_\eta \restriction \alpha = \mathcal{A}_{\eta\restriction\alpha}$*

For every first-order theory in a relational countable language (not necessarily complete), we have coded the models of T of size $\mu \leq \kappa$ in the GBS, κ^κ. In the same way we can define these structures in the GCS, 2^κ.

Definition 4.7. *Let $\omega \leq \mu \leq \kappa$ be a cardinal and T a first-order theory in a relational countable language. We define the isomorphism relation of models of size μ, $\cong_T^\mu \subseteq \kappa^\kappa \times \kappa^\kappa$, as the relation*

$$\{(\eta, \xi) | (\mathcal{A}_{\eta\restriction\mu} \models T, \mathcal{A}_{\xi\restriction\mu} \models T, \mathcal{A}_{\eta\restriction\mu} \cong \mathcal{A}_{\xi\restriction\mu}) \text{ or } (\mathcal{A}_{\eta\restriction\mu} \not\models T, \mathcal{A}_{\xi\restriction\mu} \not\models T)\}$$

Let us denote by $\cong_T$ the isomorphism relation of models of size κ of T (i.e. $\cong_T^\kappa$). To simplify notation we will refer to $\cong_T$ as the isomorphism relation of T. We will also denote by $\mathcal{A}_\eta$ the structure $\mathcal{A}_{\eta\restriction\kappa}$, for obvious reasons.

Exercise 4.2. *Let T be a first-order theory in a relational countable language. Show that the isomorphism relation of T, $\cong_T$, in the space κ^κ is continuous reducible to the isomorphism relation of T in 2^κ.*

Exercise 4.3. *Prove Proposition 4.8.*

Proposition 4.8 (Moreno, [16] Proposition 5.28). *Let $\omega < \mu < \delta \leq \kappa$ be cardinals. For all first-order countably theory in a relational countable language T, not necessarily complete,*

$$\cong_T^\mu \hookrightarrow_c \cong_T^\delta .$$

(Hint: Use Theorem 4.5 and $\kappa^{<\kappa} = \kappa$.

Exercise 4.4. *Prove 4.9.*

Proposition 4.9 (Moreno, [16] Proposition 5.30). *Let $\kappa = \aleph_\gamma$ be such that $\beth_{\omega_1}(|\gamma|) \leq \kappa$ and $\kappa = \lambda^+ = 2^\lambda$. Suppose T_1 is classifiable shallow, T_2 classifiable non-shallow, and T_3 non-classifiable. Then*

$$\cong_{T_1} \hookrightarrow_B 0_\kappa \hookrightarrow_L \cong_{T_3}^\lambda \hookrightarrow_c \cong_{T_2} .$$

(Hint: Use Theorem 4.4).

4.2 The Ehrenfeucht-Fraïssé game

Let su denote by $\mathcal{P}_\kappa(\kappa)$ the set of subsets of κ of size less than κ.

Definition 4.10 (The Ehrenfeucht-Fraïssé game). *Fix an enumeration $\{X_\gamma\}_{\gamma<\kappa}$ of the elements of $\mathcal{P}_\kappa(\kappa)$ and an enumeration $\{f_\gamma\}_{\gamma<\kappa}$ of all the functions with both the domain and range in $\mathcal{P}_\kappa(\kappa)$. For every pair of structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the $\mathrm{EF}^\alpha_\omega(\mathcal{A},\mathcal{B})$ is a game played by players* **I** *and* **II** *as follows.*

In the n-th move, first **I** *chooses an ordinal $\beta_n < \kappa$ such that $X_{\beta_{n-1}} \subseteq X_{\beta_n}$. Then* **II** *chooses an ordinal $\theta_n < \kappa$ such that $X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap \mathrm{ran}(f_{\theta_n})$ and $f_{\theta_{n-1}} \subseteq f_{\theta_n}$ (if $n = 0$ then $X_{\beta_{n-1}} = \emptyset$ and $f_{\theta_{n-1}} = \emptyset$). The game finishes after ω moves. The player* **II** *wins if $\bigcup_{i<\omega} f_{\theta_i} : A \to B$ is a partial isomorphism. Otherwise the player* **I** *wins.*

Definition 4.11 (Restricted game). *For every $\alpha \leq \kappa$ the game $\mathrm{EF}^\alpha_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ on the restrictions $\mathcal{A}\restriction\alpha$ and $\mathcal{B}\restriction\alpha$ of the structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ is defined as follows:*

In the n-th move, first **I** *chooses an ordinal $\beta_n < \alpha$ such that $X_{\beta_n} \subset \alpha$ and $X_{\beta_{n-1}} \subseteq X_{\beta_n}$. Then* **II** *chooses an ordinal $\theta_n < \alpha$ such that $dom(f_{\theta_n}), \mathrm{ran}(f_{\theta_n}) \subset \alpha$, $X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap \mathrm{ran}(f_{\theta_n})$ and $f_{\theta_{n-1}} \subseteq f_{\theta_n}$ (if $n = 0$ then $X_{\beta_{n-1}} = \emptyset$ and $f_{\theta_{n-1}} = \emptyset$). The game ends after ω moves. Player* **II** *wins if $\bigcup_{i<\omega} f_{\theta_i} : A\restriction_\alpha \to B\restriction_\alpha$ is a partial isomorphism. Otherwise player* **I** *wins. If $\alpha = \kappa$ then this is the same as the standard* EF*-game which is usually denoted by $\mathrm{EF}^\kappa_\omega$.*

We will write **I** $\uparrow \mathrm{EF}^\alpha_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ *when* **I** *has a winning strategy in the game $\mathrm{EF}^\alpha_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$. Similarly for* **II**.

Lemma 4.12 (Hyttinen-Moreno, [9] Lemma 2.4). *If $\mathcal{A}$ and $\mathcal{B}$ are structures with domain κ, then the following hold:*

- **II** $\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A},\mathcal{B}) \iff \exists C \subseteq \kappa$ *a club, such that* **II** $\uparrow \mathrm{EF}^\alpha_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ *for all $\alpha \in C$.*
- **I** $\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A},\mathcal{B}) \iff \exists C \subseteq \kappa$ *a club, such that* **I** $\uparrow \mathrm{EF}^\alpha_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ *for all $\alpha \in C$.*

Proof. It is easy to see that if $\sigma : \kappa^{<\omega} \to \kappa$ is a winning strategy for **II** in the game $\mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction\kappa, \mathcal{B}\restriction\kappa)$, then $\sigma\restriction\alpha^{<\alpha}$ is a winning strategy for **II** in the game $\mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ if $\sigma[\alpha^{<\alpha}] \subseteq \alpha$. So **II** $\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ for α a closed point of σ.

We conclude that if **II** $\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction\kappa, \mathcal{B}\restriction\kappa)$, then **II** $\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ for club-many α. The same holds for **I**. To show the other direction, notice that $\mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction\kappa, \mathcal{B}\restriction\kappa)$ is a determined game, so if **II** doesn't have a winning strategy, then **I** has a winning strategy. Therefore, if **II** doesn't have a winning strategy in the game $\mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction\kappa, \mathcal{B}\restriction\kappa)$, then **I** $\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ for club-many α, and **II** cannot have a winning strategy in $\mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ for club-many α. ☐

Definition 4.13. *Assume T is a complete first order theory in a countable vocabulary. For every $\alpha < \kappa$ and $\eta, \xi \in \kappa^\kappa$, we write $\eta\ R^\alpha_{EF}\ \xi$ if one of the following holds, $\mathcal{A}_\eta\restriction_\alpha \not\models T$ and $\mathcal{A}_\xi\restriction_\alpha \not\models T$, or $\mathcal{A}_\eta\restriction_\alpha \models T$, $\mathcal{A}_\xi\restriction_\alpha \models T$ and* **II** $\uparrow EF^\kappa_\omega(\mathcal{A}_\eta\restriction_\alpha, \mathcal{A}_\xi\restriction_\alpha)$.

Lemma 4.14 (Hyttinen-Moreno, [9] Lemma 2.7). *For every complete first order theory T in a countable vocabulary, there are club many α such that R^α_{EF} is an equivalence relation.*

Proof. Define the following functions:

- $h_1 : \kappa \to \kappa$, $h_1(\alpha) = \gamma$ where f_γ is the identity function of X_α.
- $h_2 : \kappa \to \kappa$, $h_2(\alpha) = \gamma$ where $f_\alpha^{-1} = f_\gamma$.
- $h_3 : \kappa^2 \to \kappa$, $h_3(\alpha,\beta) = X_\alpha \cup X_\beta = X_\gamma$.
- $h_4 : \kappa \to \kappa$, $h_4(\alpha) = rang(f_\alpha) = X_\gamma$.
- $h_5 : \kappa \to \kappa$, $h_5(\alpha) = dom(f_\alpha) = X_\gamma$.
- $h_6 : \kappa^2 \to \kappa$, $h_6(\alpha,\beta) = \gamma$ where $f_\alpha \circ f_\beta = f_\gamma$, $f_\alpha \circ f_\beta$ is defined on the set $f_\beta^{-1}[rang(f_\beta) \cap dom(f_\alpha)]$.

Each of these functions defines a club,

- $C_i = \{\gamma < \kappa | \forall \alpha < \gamma (h_i(\alpha) < \gamma)\}$ for $i \in \{1,2,4,5\}$.
- $C_i = \{\gamma < \kappa | \forall \beta, \alpha < \gamma (h_i(\alpha,\beta) < \gamma)\}$ for $i \in \{3,6\}$.

Denote by C the club $\cap_{i=1}^6 C_i$. We will show that for every $\alpha \in C$, R^α_{EF} is an equivalence relation.

By definition $\eta\ R^\alpha_{EF}\ \xi$ implies that either both $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ are models of T or non of them is a model of T. Thus $R^\alpha_{EF} = R^- \cup R^+$, where R^- is the restriction of R^α_{EF} to the set $A = \{\eta \in \kappa | \mathcal{A}_\eta \not\models T\}$ and R^+ is the restriction of R^α_{EF} to the complement of A. Since $R^- \cap R^+ = \emptyset$, it is enough to prove that R^- and R^+ are equivalence relations.
By definition it is easy to see that $R^- = A \times A$, therefore R^- is an equivalence relation. Now we will prove that R^+ is an equivalence relation.

Reflexivity
By the way C_1 was defined, for every $\beta < \alpha$, $h_1(\beta) < \alpha$ and $f_{h_1(\beta)}$ is the identity function of X_β. Therefore, the function $\sigma((\beta_0, \beta_1, \dots, \beta_n)) = h_1(\beta_n)$ is a winning strategy for **II** in the game $\mathrm{EF}^\kappa_\omega(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_\eta \restriction_\alpha)$.

Symmetry
Let σ be a winning strategy for **II** in the game $\mathrm{EF}^\kappa_\omega(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_\xi \restriction_\alpha)$. Since $\alpha \in C_2$ and $\sigma((\beta_0, \beta_1, \dots, \beta_n)) < \alpha$, we know that $h_2(\sigma((\beta_0, \beta_1, \dots, \beta_n))) < \alpha$. Notice that if $\cup_{i<\omega} f_{\theta_i} : \alpha \to \alpha$ is a partial isomorphism from $\mathcal{A}_\eta \restriction_\alpha$ to $\mathcal{A}_\xi \restriction_\alpha$, then $\cup_{i<\omega} f_{h_2(\theta_i)} = \cup_{i<\omega} f^{-1}_{\theta_i}$ is a partial isomorphism from $\mathcal{A}_\xi \restriction_\alpha$ to $\mathcal{A}_\eta \restriction_\alpha$. Therefore, the function $\sigma'((\beta_0, \beta_1, \dots, \beta_n)) = h_2(\sigma((\beta_0, \beta_1, \dots, \beta_n)))$ is a winning strategy for **II** in the game $\mathrm{EF}^\kappa_\omega(\mathcal{A}_\xi \restriction_\alpha, \mathcal{A}_\eta \restriction_\alpha)$.

Transitivity
Let σ_1 and σ_2 be two winning strategies for **II** on the games $\mathrm{EF}^\kappa_\omega(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_\xi \restriction_\alpha)$ and $\mathrm{EF}^\kappa_\omega(\mathcal{A}_\xi \restriction_\alpha, \mathcal{A}_\zeta \restriction_\alpha)$, respectively.
For a given tuple $(\beta_0, \beta_1, \dots, \beta_n)$ let us construct by induction the tuples $(\gamma_0, \gamma_1, \dots, \gamma_n)$, $(\beta'_0, \beta'_1, \dots, \beta'_{2n}, \beta'_{2n+1})$, and the functions $f_{(1,n)}$, g_n and $f_{(2,n)}$:

1. Let $\beta'_0 = \beta_0$ and for $i > 0$, let β'_{2i} be the least ordinal such that $X_{\beta'_{2i-1}} \cup X_{\beta_i} = X_{\beta'_{2i}}$.
2. $f_{(1,i)} := f_{\sigma_1((\beta'_0, \beta'_1, \dots, \beta'_{2i-1}, \beta'_{2i}))}$.
3. γ_i is the ordinal such that $X_{\gamma_i} = rang(f_{(1,i)})$.
4. $g_i := f_{\sigma_2((\gamma_0, \gamma_1, \dots, \gamma_i))}$.
5. β'_{2i+1} is the ordinal such that $X_{\beta'_{2i+1}} = dom(g_i)$.
6. $f_{(2,i)} := f_{\sigma_1((\beta'_0, \beta'_1, \dots, \beta'_{2i}, \beta'_{2i+1}))}$.

Define the function $\sigma : \alpha^{<\omega} \to \alpha$ by $\sigma((\beta_0, \beta_1, \dots, \beta_n)) = \theta_n$, where θ_n is the ordinal such that $f_{\theta_n} = g_n \circ (f_{(2,n)} \restriction_{f^{-1}_{(2,n)}[dom(g_n)]})$. It is easy to check that for every n, the tuples $(\gamma_0, \gamma_1, \dots, \gamma_n)$ and $(\beta'_0, \beta'_1, \dots, \beta'_{2n+1})$ are elements of $\alpha^{<\omega}$, and the functions $f_{(1,n)}$, g_n, $f_{(2,n)}$ and f_{θ_n} are well defined; it is also easy to check that $\sigma((\beta_0, \beta_1, \dots, \beta_n))$ is a valid move.
Let us show that $\cup_{n<\omega} f_{\theta_n}$ is a partial isomorphism. It is clear that $rang(f_{(2,n)}) \subseteq rang(f_{(1,n+1)})$. By 3 and 4 in the induction, we can conclude that $rang(f_{(2,n)})$ is a subset of $dom(g_{n+1})$. Then $rang(\cup_{n<\omega}(f_{(2,n)})) \subseteq dom(\cup_{n<\omega}(g_n))$, so

$$\cup_{n<\omega}(g_n \circ (f_{(2,n)} \restriction_{f^{-1}_{(2,n)}[dom(g_n)]})) = \cup_{n<\omega}(g_n) \circ \cup_{n<\omega}(f_{(2,n)}).$$

Since σ_1 and σ_2 are winning strategies, we know that $\cup_{n<\omega}(g_n)$ and $\cup_{n<\omega}(f_{(2,n)})$ are partial isomorphism. Therefore $\cup_{n<\omega} f_{\theta_n}$ is a partial isomorphism and σ is a winning strategy for **II** on the game $\mathrm{EF}^\kappa_\omega(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_\zeta \restriction_\alpha)$. □

Corollary 4.15. *Suppose $\eta, \xi \in \kappa^\kappa$. Then the following hold:*

- $\eta\ R^\kappa_{EF}\ \xi \iff \exists C \subseteq \kappa$ *a club, such that* $\eta\ R^\alpha_{EF}\ \xi$ *for all* $\alpha \in C$.
- $\neg(\eta\ R^\alpha_{EF}\ \xi) \iff \exists C \subseteq \kappa$ *a club, such that* $\neg(\eta\ R^\alpha_{EF}\ \xi)$ *for all* $\alpha \in C$.

4.3 Classifiable theories

The reason to introduce these games is that we can characterize classifiable theories with these games.

Theorem 4.16 (Shelah, [19], XIII Theorem 1.4)**.** *If T is a classifiable theory, then every two models of T that are $L_{\infty,\kappa}$-equivalent are isomorphic.*

Theorem 4.17 (Friedman-Hyttinen-Weinstein(Kulikov), [5], Theorem 10)**.** *$L_{\infty,\kappa}$-equivalence is equivalent to EF^κ_ω-equivalence.*

From these two theorems we know that if T is a classifiable theory, then for any $\mathcal{A}$ and $\mathcal{B}$ models of T with domain κ,

$$\mathbf{II}\uparrow \mathrm{EF}^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \iff \mathcal{A}\cong\mathcal{B}$$

$$\mathbf{I}\uparrow \mathrm{EF}^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \iff \mathcal{A}\not\cong\mathcal{B}.$$

Theorem 4.18 (Friedman-Hyttinen-Weinstein(Kulikov), [5], Theorem 70)**.** *If T is a classifiable theory, then $\cong_T$ is $\Delta^1_1(\kappa)$.*

Proof. Notice that the $\mathrm{EF}^{\kappa}_{\omega}$ game can be coded as a κ-Borel* game taking at the leaves the open sets given by partial isomorphisms. □

From Lemma 4.12, we know the following two hold for any $\mathcal{A}$ and $\mathcal{B}$ models of a classifiable theory (with domain κ):

- $\mathcal{A}\cong\mathcal{B} \iff \mathbf{II}\uparrow \mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α.
- $\mathcal{A}\not\cong\mathcal{B} \iff \mathbf{I}\uparrow \mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α.

Clearly R^{κ}_{EF} coincide with $\cong_T$ when T is classifiable. So

- $\eta\ \cong^{\kappa}_T\ \xi \iff \exists C\subseteq\kappa$ a club, such that $\eta\ R^{\alpha}_{EF}\ \xi$ for all $\alpha\in C$.
- $\neg(\eta\ \cong^{\alpha}_T\ \xi) \iff \exists C\subseteq\kappa$ a club, such that $\neg(\eta\ R^{\alpha}_{EF}\ \xi)$ for all $\alpha\in C$.

Theorem 4.19 (Hyttinen-Moreno, [9] Theorem 2.8)**.** *Assume T is a countable complete classifiable theory over a countable vocabulary, $S\subseteq\kappa$ a stationary set, and μ a regular cardinal. Then $\cong^{\kappa}_T\ \hookrightarrow_L\ =^{\kappa}_S$.*

Proof. It follows from the approximation lemma (Lemma 2.19), Lemma 4.14, and Lemma 4.12. □

Exercise 4.5. *Prove Theorem 4.20.*

Theorem 4.20 (Hyttinen-Weisnstein(Kulikov)-Moreno, [7] Lemma 2)**.** *Assume T is a countable complete classifiable theory over a countable vocabulary. Let $S\subseteq\kappa$ a stationary set. If $\diamondsuit_S$ holds, then $\cong^{\kappa}_T\ \hookrightarrow_L\ =^2_S$.*

5 Further results

5.1 Borel sets, Δ^1_1 sets, Borel* sets and Σ^1_1 sets

Theorem 5.1 (Hyttinen-Weisnstein(Kulikov), [6], Corollary 3.2)**.** *It is consistent that $\Delta^1_1(\kappa)\subsetneq\kappa$-Borel* $\subsetneq\Sigma^1_1(\kappa)$.*

Lemma 5.2 (Friedman-Hyttinen-Weinstein(Kulikov), [5], Corollary 14)**.** *The set $\{(\eta,\xi)\in\kappa^\kappa\times\kappa^\kappa \mid \mathcal{A}_\eta\cong\mathcal{A}_\xi\}$ is $\Sigma^1_1(\kappa)$.*

Theorem 5.3 (Friedman-Hyttinen-Weinstein(Kulikov), [5], Theorem 24)**.** *A set $B\subseteq\kappa^\kappa$ is κ-Borel and closed under permutations if and only if there is a sentence φ in $L_{\kappa^+\kappa}$ such that $B=\{\eta\in\kappa^\kappa\mid\mathcal{A}_\eta\models\varphi\}$.*

Theorem 5.4 (Friedman-Hyttinen-Kulikov)**.**

1. *Let $\kappa^{<\kappa}=\kappa>2^\omega$. If T is classifiable and shallow, then $\cong_T$ is κ-Borel.* ([5], Theorem 68)
2. *If T is classifiable non-shallow, then $\cong_T$ is $\Delta^1_1(\kappa)$ not κ-Borel.* ([5], Theorem 69 and 70)
3. *If T is unstable or stable with the OTOP or superstable with the DOP and $\kappa>\omega_1$, then $\cong_T$ is not $\Delta^1_1(\kappa)$.* ([5], Theorem 71)
4. *If T is stable unsuperstable, then $\cong_T$ is not κ-Borel.* ([5], Theorem 72)

5.2 Non-reducible results

Theorem 5.5 (Friedman-Hyttinen-Weinstein(Kulikov), [5], Theorem 52). *Assume GCH, $\mu < \kappa$ a regular cardinal such that if $\kappa = \lambda^+$, then $\mu \leq cf(\lambda)$. Then in a cofinality and GCH preserving forcing extension, there stationary sets $K(A) \subseteq S^\kappa_\mu$ for each $A \subseteq \kappa$ such that $=^\kappa_{K(A)} \not\hookrightarrow_B =^\kappa_{K(B)}$ if and only if $A \not\subseteq B$.*

Theorem 5.6 (Friedman-Hyttinen-Weinstein(Kulikov), [5], Theorem 56). *For a cardinal κ which is a successor of a regular cardinal or it is inaccessible, there is a cofinality-preserving forcing extension in which for all regular $\lambda < \kappa$, the relations $=^\kappa_\lambda$ are $\hookrightarrow_B$-incomparable with each other.*

Theorem 5.7 (Dense non-reduction; Fernandes-Moreno-Rinot, [3] Corollary 6.19). *There exists a cofinality-preserving forcing extension in which:*

- *For all stationary subsets X, S of S, there exist stationary subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that $=^2_{X'} \not\hookrightarrow_B =^\kappa_{Y'}$.*
- *For all two disjoint stationary subsets X, Y of κ, $=^2_X \not\hookrightarrow_B =^\kappa_Y$.*

Theorem 5.8 (Friedman-Hyttinen-Weinstein(Kulikov), [5] Theorem 77). *If a first order countable complete theory over a countable vocabulary T is classifiable, then $=^2_\omega \not\hookrightarrow_c \cong_T$.*

5.3 Reflections

Theorem 5.9 (Shelah, [20] Claim 2.3). *For an uncountable cardinal λ, and a stationary subset $S \subseteq S^{\lambda^+}_{\neq cf(\lambda)}$, the following are equivalent:*

- $2^\lambda = \lambda^+$,
- $\diamondsuit_{\lambda^+}(S)$.

Definition 5.10. *For a stationary $S \subseteq \kappa$, $\diamondsuit^{++}_S$ asserts the existence of a sequence $\langle K_\alpha \mid \alpha \in S \rangle$ satisfying the following:*

1. *for every infinite $\alpha \in S$, K_α is a set of size $|\alpha|$;*
2. *for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $C \cap \alpha, X \cap \alpha \in K_\alpha$;*
3. *the following set is stationary in $[H_{\kappa^+}]^{<\kappa}$:*

$$\{M \in [H_{\kappa^+}]^{<\kappa} \mid M \cap \kappa \in S \ \&\ \text{clps}(M, \in) = (K_{M\cap\kappa}, \in)\}.$$

Theorem 5.11 (Sakai, [18] Prop 1.4). *$\diamondsuit^{++}_S$ holds in L.*

Lemma 5.12 (Fernandes-Moreno-Rinot, [3], Thm 4.10). *For every stationary $S \subseteq \kappa$, $\diamondsuit^{++}_S$ implies $\text{Dl}^*_S(\Pi^1_2)$.*

Definition 5.13. *Let $\mathbb{S}$ be the poset of all pairs $(k, \mathcal{B})$ with the following properties:*

1. *k is a function such that $dom(k) < \kappa$;*
2. *for each $\alpha \in dom(k), k(\alpha)$ is a transitive model of ZF^- of size $\leq \max\{\aleph_0, |\alpha|\}$, with $k \restriction \alpha \in k(\alpha)$;*
3. *$\mathcal{B}$ is a subset of $\mathcal{P}(\kappa)$ of size $\leq \text{dom}(k)$;*

$(k', \mathcal{B}') \leq (k, \mathcal{B})$ in $\mathbb{S}$ if the following holds:

(i) $k' \supseteq k$, and $\mathcal{B}' \supseteq \mathcal{B}$;

(ii) for any $B \in \mathcal{B}$ and any $\alpha \in dom(k') \setminus dom(k)$, $B \cap \alpha \in k'(\alpha)$.

Lemma 5.14 (Sakai, [18] Prop 1.5). *For every stationary $S \subseteq \kappa$, $V^{\mathbb{S}} \models \diamondsuit^{++}_S$.*

Corollary 5.15 (Fernandes-Moreno-Rinot, [3] Lemma 4.10 and Proposition 4.14). *There exists a $<\kappa$-closed κ^+-cc forcing extension in which $\text{Dl}^*_{\check{S}}(\Pi^1_2)$ holds for all $\check{S} \subseteq \kappa$ stationary set (S stationary in V).*

Since $\diamondsuit^{++}_S$ holds in L, in L we have κ-Borel* $= \Sigma^1_1(\kappa)$. Also there is a $<\kappa$-closed κ^+-cc forcing which forces κ-Borel* $= \Sigma^1_1(\kappa)$.

Definition 5.16. *For a given cardinal $\lambda = \mu^+$ and a stationary set $S \subseteq \lambda$, $\diamondsuit^+_S$ is the statement that there is a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in S \rangle$ such that*

- *For all $\alpha \in S$, $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$ and $|\mathcal{A}_\alpha| \leq \mu$.*
- *If $Z \subseteq \lambda$, then there exists a club $C \subseteq \lambda$ such that*

$$C \cap S \subseteq \{\alpha \in S \mid Z \cap \alpha \in \mathcal{A}_\alpha \ \& \ C \cap \alpha \in \mathcal{A}_\alpha\}.$$

Lemma 5.17 (Fernandes-Moreno-Rinot, [3] Corollary 4.12)**.** *It is consistent that $\diamondsuit_S^+$ holds, but $\diamondsuit_S^{++}$ fails.*

Theorem 5.18 (Fernandes-Moreno-Rinot, [3] Corollary 5.7)**.** *If κ is strongly inaccessible, then in the forcing extension by $Add(\kappa, \kappa^+)$, for all stationary subsets X, S of κ, the following are equivalent:*

1. *X $\mathfrak{f}$-reflects to S;*
2. *every stationary subset of X reflects in S.*

Theorem 5.19 (Fernandes-Moreno-Rinot, [3] Corollary 5.12)**.** *There exists a cofinality-preserving forcing extension in which, for all stationary subsets X, S of κ, X does not $\mathfrak{f}$-reflects to S.*

5.4 Model theory

The smallest ordinal α such that $A \in \Sigma^0_\alpha \cup \Pi^0_\alpha$ is called the Borel rank of A and denoted by $rk_B(A)$. Given a theory T, let us denote by $B(\kappa, T)$ the rank $rk_B(\cong_T)$.

Theorem 5.20 (Descriptive Main Gap; Mangraviti-Motto Ros, [13] Theorem 1.9)**.** *Let $\kappa > 2^\omega$. If T is classifiable shallow of depth α, then $B(\kappa, T) \leq 4\alpha$.*

A theory T is κ-categorical if there is only one model of T of size κ up to isomorphism. A theory T is categorical in κ if T is κ-categorical.

Theorem 5.21 (Morley's categoricity theorem, [17] Theorem 5.6)**.** *Let T be a first-order countable complete theory. If T is categorical in one uncountable cardinal, then T is categorical in every uncountable cardinal.*

Theorem 5.22 (Mangraviti-Motto Ros, [13] Theorem 3.3)**.** *Let T be a countable first-order theory in a countable vocabulary (not necessarily complete). T is κ-categorical if and only if $rk_B(\cong_T) = 0$, i.e. $\cong_T$ is clopen.*

Theorem 5.23 (Strictly stable; Hyttinen-Kulikov-Moreno, [7] Corollary 2)**.** *Suppose that $\kappa = \lambda^+$ and $\lambda^\omega = \lambda$. If T_1 is a classifiable theory and T_2 is a stable unsuperstable theory, then $\cong_{T_1} \hookrightarrow_c \cong_{T_2}$ and $\cong_{T_2} \not\hookrightarrow_B \cong_{T_1}$.*

Theorem 5.24 (Unsuperstable; Moreno, [15] Corollary 4.12)**.** *Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^\omega = \lambda$. If T_1 is a classifiable theory, and T_2 is an unsuperstable theory, then $\cong_{T_1} \hookrightarrow_c \cong_{T_2}$ and $\cong_{T_2} \not\hookrightarrow_B \cong_{T_1}$.*

Theorem 5.25 (Borel reducibility Main Gap; Moreno, [16] Theorem 5.5)**.** *Let $\mathfrak{c} = 2^\omega$. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $2^\mathfrak{c} \leq \lambda = \lambda^{\omega_1}$. If T_1 is a countable complete classifiable shallow theory, T_2 is a countable complete classifiable theory not shallow, and T_3 is a countable complete non-classifiable theory, then the following hold:*

1. **Classifiable vs Non-classifiable.** *For $T = T_1, T_2$ there is $\gamma < \kappa$ such that:*

$$\cong_T \hookrightarrow_c =^2_\gamma \hookrightarrow_c \cong_{T_3} \ \text{ and } \ \cong_{T_3} \not\hookrightarrow_B \cong_T .$$

2. **Shallow vs Non-shallow.** *If $\kappa = \aleph_\mu$ is such that $\beth_{\omega_1}(|\ \mu\ |) \leq \kappa$, then*

$$\cong_{T_1} \hookrightarrow_B \ 0_\kappa \ \hookrightarrow_B \cong_{T_2} \hookrightarrow_c \cong_{T_3} .$$

In particular,

$$\cong_{T_3} \not\hookrightarrow_B \cong_{T_2} \not\hookrightarrow_r \ 0_\kappa \ \not\hookrightarrow_r \cong_{T_1} .$$

Theorem 5.26 (L-Main Gap Dichotomy; Hyttinen-Kulikov-Moreno, [8] Theorem 4.11)**.** *($V = L$). Suppose $\kappa = \lambda^+$ and λ is a regular uncountable cardinal. If T is a countable first-order theory in a countable vocabulary, not necessarily complete, then one of the following holds:*

- *$\cong_T$ is $\Delta^1_1(\kappa)$.*
- *$\cong_T$ is $\Sigma^1_1(\kappa)$-complete.*

Theorem 5.27 (Main Gap Dichotomy; Moreno, [16] Theorem 5.16)**.** *Let κ be inaccessible, or $\kappa = \lambda^+ = 2^\lambda$ and $2^\mathfrak{c} \leq \lambda = \lambda^{<\omega_1}$. There exists a $< \kappa$-closed κ^+-cc forcing extension in which for any countable first-order theory in a countable vocabulary (not necessarily complete), T, one of the following holds:*

- *$\cong_T$ is $\Delta^1_1(\kappa)$.*
- *$\cong_T$ is $\Sigma^1_1(\kappa)$-complete.*

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