Series of lectures on Generalized Descriptive Set Theory

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March 10, 2019

This notes are based on a series of talks given at the Set Theory seminar of Bar-Ilan University. This notes are intended to be as close as possible to the transcripts of those seminar session. Due to the nature of the seminar and the questions from the audience, some proofs were split into different sessions in order to give examples and clear answers to the questions from the audience.

1 An introduction to generalized descriptive set theory

Generalized descriptive set theory is the generalization of descriptive set theory to uncountable cardinals. For a background on classical descriptive set theory see [7] or [8]. During this notes, κ will be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$, unless otherwise is stated.

First Session

The aim of this first section is to introduce the notions of κ-Borel class, $\Delta_1^1(\kappa)$ class, κ-Borel* class, and show the relation between these classes.

Definition 1.1 (The Generalized Baire space $\mathbf{B}(\kappa)$)**.** *Let κ be an uncountable cardinal. The generalized Baire space is the set κ^κ endowed with the following topology. For every $\eta \in \kappa^{<\kappa}$, define the following basic open set*

$$N_\eta = \{f \in \kappa^\kappa \mid \eta \subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

Definition 1.2 (The Generalized Cantor space $\mathbf{C}(\kappa)$)**.** *Let κ be an uncountable cardinal. The generalized Cantor space is the set 2^κ endowed with the following topology. For every $\eta \in 2^{<\kappa}$, define the following basic open set*

$$N_\eta = \{f \in 2^\kappa \mid \eta \subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

Definition 1.3 (κ-Borel class)**.** *Let $S \in \{\mathbf{B}(\kappa), \mathbf{C}(\kappa)\}$. The class κ-Borel(S) of all κ-Borel sets in S is the least collection of subsets of S which contains all open sets and is closed under complements, unions and intersections both of length at most κ.*

Definition 1.4. *Let $S \in \{\mathbf{B}(\kappa), \mathbf{C}(\kappa)\}$.*

- *$X \subset S$ is a $\Sigma_1^1(\kappa)$ set if there is a set $Y \subset S \times S$ a closed set such that $pr(Y) = \{x \in S \mid \exists y \in S\ (x, y) \in Y\} = X$.*
- *$X \subset S$ is a $\Pi_1^1(\kappa)$ set if $S\backslash X$ is a $\Sigma_1^1(\kappa)$ set.*
- *$X \subset S$ is a $\Delta_1^1(\kappa)$ set if X is a $\Sigma_1^1(\kappa)$ set and a $\Pi_1^1(\kappa)$ set.*

Definition 1.5 (κ-*Borel**-set in $\mathbf{B}(\kappa), \mathbf{C}(\kappa)$)**.** *Let $S \in \{2^\kappa, \kappa^\kappa\}$.*

1. *A subset $T \subset \kappa^{<\kappa}$ is a tree if for all $f \in T$ with $\alpha = dom(f) > 0$ and for all $\beta < \alpha$, $f \restriction \beta \in T$ and $f \restriction \beta < f$.*
2. *A tree T is a κ^+, λ-tree if does not contain chains of length λ and its cardinality is less than κ^+. It is closed if every chain has a unique supremum in T.*
3. *A pair (T, h) is a κ-Borel*-code if T is a closed κ^+, λ-tree, $\lambda \leq \kappa$, and h is a function with domain T such that if $x \in T$ is a leaf, then $h(x)$ is a basic open set and otherwise $h(x) \in \{\cup, \cap\}$.*

4. *For an element $\eta \in S$ and a κ-Borel*-code (T, h), the κ-Borel*-game $B^*(T, h, \eta)$ is played as follows. There are two players,* **I** *and* **II**. *The game starts from the root of T. At each move, if the game is at node $x \in T$ and $h(x) = \cap$, then* **I** *chooses an immediate successor y of x and the game continues from this y. If $h(x) = \cup$, then* **II** *makes the choice. At limits the game continues from the (unique) supremum of the previous moves. Finally, if $h(x)$ is a basic open set, then the game ends, and* **II** *wins if and only if $\eta \in h(x)$.*

5. *A set $X \subseteq S$ is a κ-Borel*-set if there is a κ-Borel*-code (T, h) such that for all $\eta \in S$, $\eta \in X$ if and only if* **II** *has a winning strategy in the game $B^*(T, h, \eta)$.*

We will write $\mathbf{II} \uparrow B^*(T, h, \eta)$ when **II** has a winning strategy in the game $B^*(T, h, \eta)$.

Example 1.1. *Let $\mu < \kappa$ be a regular cardinal, we say that $X \subseteq \kappa$ is a μ-club if X is an unbounded set and it is closed under μ-limits.*

Let $\mu < \kappa$ be a regular cardinal. For all $\eta, \xi \in 2^\kappa$ we say that η and ξ are $E^2_{\mu\text{-club}}$ equivalent if the set $\{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\}$ contains a μ-club.

The relation $E^2_{\omega\text{-club}}$ is a κ-Borel* set. Let us define the following κ-Borel*-code (T, h):

- *$T = \{f \in \kappa^{<\omega+2} \mid f$ is strictly incresing$\}$.*
- *For f not a leave, $h(f) = \cup$ if $dom(f)$ is even and $h(f) = \cap$ if $dom(f)$ is odd.*
- *To define $h(f)$ for a leave f, first define the set $L(g) = \{f \in \kappa^{\omega+1} \mid g \subseteq f\}$ for all $g \in T$ with domain ω, and $\gamma_g = sup_{n<\omega}(g(n))$. Let $h \restriction L(g)$ be a bijection between $L(g)$ and the set $\{N_p \times N_q \mid p, q \in \kappa^{\gamma_g+1}, p(\gamma_g) = q(\gamma_g)\}$.*

Let us show that (T, h) codes $E^2_{\omega\text{-club}}$. Suppose $(\eta, \xi) \in E^2_{\omega\text{-club}}$, so there is an ω-club C such that $\forall \alpha \in C$ $\eta(\alpha) = \xi(\alpha)$. The following is a winning strategy for **II** *in the game $B^*(T, h, (\eta, \xi))$. For every even $n < \omega$, if the game is at f with $dom(f) = n$,* **II** *chooses an immediate successor f' of f, such that $f \subset f'$ and $f'(n) \in C$. Since C is closed under ω limits, after ω moves the game continues at $g \in \kappa^\omega$ strictly increasing with $\gamma = sup_{n<\omega}(g(n)) \in C$. So there is G an immediate successor of g, such that $h(G) = N_{\eta\restriction\gamma} \times N_{\xi\restriction\gamma}$. Finally if* **II** *chooses G in the ω move, then* **II** *wins.*

For the other direction, suppose $(\eta, \xi) \notin E^2_{\omega\text{-club}}$, so there is $A \subset S^\kappa_\omega$ stationary (S^κ_ω is the set of ω-cofinal ordinals below κ) such that for all $\alpha \in S$, $\eta(\alpha) \neq \xi(\alpha)$.

We will show that for every σ strategy of **II**, *σ is not a winning strategy. Let σ be an strategy for* **II**, *this mean that σ is a function from $\kappa^{<\omega+1} \to \kappa$. Notice that if* **II** *follows σ as a strategy, then when the game is at f, $dom(f) = n$ even,* **II** *chooses f' such that $f \subset f'$ and $f'(n) = \sigma((f(0), f(1), \dots, f(n-1)))$. Let C be the set of closed points of σ, $C = \{\alpha < \kappa \mid \sigma(\alpha^{<\omega}) \subseteq \alpha\}$, C is unbounded and closed under ω-limits. Therefore $C \cap A \neq \emptyset$. Let γ be the least element of $C \cap A$ that is an ω-limit of elements of C, and let $\{\gamma_n\}_{n<\omega}$ be a sequence of elements of C cofinal to γ. The following is a winning strategy for* **I** *in the game $B^*(T, h, (\eta, \xi))$, if* **II** *uses σ as an strategy.*

When the game is at f with $dom(f) = n$, n odd, then **I** *chooses an immediate successor f' of f, such that $f \subset f'$ and $f'(n)$ is the least element of $\{\gamma_n\}_{n<\omega}$ that is bigger than $f(n-1)$. This element always exists because $\{\gamma_n\}_{n<\omega}$ is cofinal to γ and $\gamma \in C$, γ is a closed point of σ. Since* **I** *is following σ as a strategy and γ is a closed point of σ, after ω moves the game continues at $g \in \kappa^\omega$ strictly increasing with $\gamma = sup_{n<\omega}(g(n)) \in C \cap A$. Since $\eta(\gamma) \neq \xi(\gamma)$, there is no G immediate successor of g, such that $(\eta, \xi) \in h(G)$. So it does not matter what* **II** *chooses in the ω move,* **I** *will win.*

The previous definitions are the generalization of the notions of Borel, Δ^1_1, and Borel* from descriptive set theory, the spaces ω^ω and 2^ω. A classical result in descriptive set theory states that the Borel class, the Δ^1_1 class, and the Borel* class are the same. This doesn't hold in generalized descriptive set theory as we will see.

Theorem 1.6 ([2], Thm 17). *κ-Borel$\subseteq$ κ-Borel**

Proof. Let us prove something even stronger. X is a κ-Borel set if and only if there is a κ-Borel*-code (T, h) such that (T, h) codes X and T is a κ^+, ω-tree.

Let us define the sets $(B_i)_{i\leq\kappa^+}$ by:

- $B_0 = \{N_p \mid p \in 2^{<\kappa}\}$, the set of basic open sets.
- If $\alpha = \beta + n$ for n an odd natural number and β a limit ordinal or 0, then $B_\alpha = B_{\beta+n-1} \cup \{\bigcap \mathcal{B} \mid \mathcal{B} \subseteq B_{\beta+n-1}, \mid \mathcal{B} \mid \leq \kappa\}$.
- If $\alpha = \beta + n$ for n an even positive natural number and β a limit ordinal or 0, then $B_\alpha = B_{\beta+n-1} \cup \{\bigcup \mathcal{B} \mid \mathcal{B} \subseteq B_{\beta+n-1}, \mid \mathcal{B} \mid \leq \kappa\}$.

- If α is a limit ordinal, then $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$.

We will show by induction over α that for every $X \in B_\alpha$, there is a κ-Borel*-code (T,h) such that (T,h) codes X and T is a κ^+,ω-tree.

For $\alpha = 0$. If $X \in B_0$, then $T = \{\emptyset\}$ and $h(\emptyset) = X$ is a κ-Borel*-code that codes X.

Suppose $\alpha = \beta + n$ for n an even natural number and β a limit ordinal or 0 is such that for all $X \in B_\alpha$, there is a κ-Borel*-code (T,h) such that (T,h) codes X and T is a κ^+,ω-tree. Suppose $X \in B_{\alpha+n+1}$, so either $X \in B_\alpha + n$ or $X = \bigcap \mathcal{B}$ for some $\mathcal{B} \subseteq B_{\beta+n}$ with $|\mathcal{B}| = \gamma \leq \kappa$. Let $\mathcal{B} = \{X_i\}_{i<\gamma}$, by the induction hypothesis we know that there are κ-Borel*-code $\{(T_i,h_i)\}_{i<\gamma}$ such that (T_i,h_i) codes X_i and T_i is a κ^+,ω-tree, for all $i<\gamma$. Let $\mathcal{T} = \{r\} \cup \bigcup_{i<\gamma} T_i \times \{i\}$ be the tree ordered by $r < (x,j)$ for all $(x,j) \in \bigcup_{i<\gamma} T_i \times \{i\}$, and $(x,i) < (y,j)$ if and only if $i = j$ and $x < y$ in T_i. Let $T \subseteq \kappa^{<\omega}$ be a tree isomorphic to $\mathcal{T}$ and let $\mathcal{G} : T \to \mathcal{T}$ be a tree isomorphism. If $\mathcal{G}(x) \neq r$, then denote $\mathcal{G}(x)$ by $(\mathcal{G}_1(x), \mathcal{G}_2(x))$. Define h by $h(x) = \cap$ if $G(x) = r$, and $h(x) = h_{\mathcal{G}_2(x)}(\mathcal{G}_1(x))$.

Let us show that (T,h) codes X. Let $\eta \in X$, so $\eta \in X_i$ for all $i < \gamma$. If at the beginning **I** chooses x, then **II** follows the winning strategy from the game $B^*(T_{\mathcal{G}_2(x)}, h_{\mathcal{G}_2(x)}, \eta)$, choosing the element given by $\mathcal{G}^{-1}$. We conclude that **II** $\uparrow B^*(T,h,\eta)$. Let $\eta \notin X$, so there is $i < \gamma$ such that $\eta \notin X_i$, so **II** has no winning strategy for the game $B^*(T_i,h_i,\eta)$. Since at the beginning **I** can choose x such that $\mathcal{G}_2(x) = i$, **II** cannot have a winning strategy for the game $B^*(T,h,\eta)$. Otherwise **II** would have a winning strategy the game $B^*(T_i,h_i,\eta)$.

The case $\alpha = \beta + n$ for n an odd natural number and β a limit ordinal or 0 is similar, just make $h(x) = \cup$ if $G(x) = r$ when constructing (T,h).

Suppose α is a limit ordinal such that for all $\beta < \alpha$, for all $X \in B_\beta$, there is a κ-Borel*-code (T,h) such that (T,h) codes X and T is a κ^+,ω-tree. Let $X \in B_\alpha$, since $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ there is $\beta < \alpha$ such that $X \in B_\beta$. By the induction hypothesis, there is a κ-Borel*-code (T,h) such that (T,h) codes X and T is a κ^+,ω-tree. □

Second Session

Theorem 1.7 ([2], Thm 17). *1. κ-Borel* $\subseteq \Sigma^1_1(\kappa)$.*

2. κ-Borel$\subseteq \Sigma^1_1(\kappa)$.

3. κ-Borel$\subseteq \Delta^1_1(\kappa)$.

Proof. 1. Let X be a κ-Borel* set, there is a κ-Borel* code (T,h) such that X is coded by (T,h).

Since $\kappa^{<\kappa} = \kappa$, we can code the strategies $\sigma : T \to T$ by elements of κ^κ.

Claim 1.8. *The set $Y = \{(\eta,\xi) \mid \xi$ is a code of a winning strategy for* **II** *in $B^*(T,h,\eta)\}$ is closed.*

Proof. Let (η,ξ) be an element not in Y. So ξ is not a winning strategy for **II** in $B^*(T,h,\eta)\}$, there is $\alpha < \kappa$ such that for every $\zeta \in N_{\xi\restriction\alpha}$, ζ is not a winning strategy for **II** in $B^*(T,h,\eta)\}$. Otherwise T would have a branch of length κ. Because of the same reason, there is $\beta < \kappa$ such that for every $f \in N_{\eta\restriction\beta}$, $\zeta \in N_{\xi\restriction\alpha}$, ζ is not a winning strategy for **II** in $B^*(T,h,f)\}$. So $N_{\eta\restriction\beta} \times N_{\xi\restriction\alpha}$ is a subset of the complement of Y. □

Since $pr(Y) = X$, we are done.

2. It follows from Theorem 1.6 and (1).

3. It follows from (2) and the fact that κ-Borel sets are closed under complement.

□

It has been proved, under the assumption $V = L$, that κ-Borel* $= \Sigma^1_1(\kappa)$. It was first proved in [2] Theorem 18, the idea of this proof is to show that the filter of ω-clubs is $\Sigma^1_1(\kappa)$-complete and κ-Borel*. This result was later improve in [3] Theorem 7 to show that the relations $E^\kappa_{\mu\text{-club}}$ is $\Sigma^1_1(\kappa)$-complete and κ-Borel*, where $\eta,\xi \in \kappa^\kappa$ are $E^\kappa_{\mu\text{-club}}$ related if the set $\{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\}$ contains an μ-club. Recently in [5] Theorem 3.1 these results were improve to show that the inclusion modulo the non-stationary ideal (below) is $\Sigma^1_1(\kappa)$-complete, which implies that the relations $E^2_{\mu\text{-club}}$ are $\Sigma^1_1(\kappa)$-complete. Because of its applications for future sessions, we will prove (under the assumption $(V = L)$) that the inclusion modulo the non-stationary ideal is $\Sigma^1_1(\kappa)$-complete, this will implies the consistency of κ-Borel* $= \Sigma^1_1(\kappa)$.

Definition 1.9 (Inclusion modulo non-stationaries). *For $\eta,\xi \in 2^\kappa$ and a stationary $S \subset \kappa$, we write $\eta \sqsubseteq_S \xi$ if $(\eta^{-1}\{1\} \backslash \xi^{-1}\{1\}) \cap S$ is non-stationary. If $S = S^\kappa_\mu$, we denoted $\sqsubseteq_S$ by $\sqsubseteq_\mu$.*

If Q_1 and Q_2 are quasi-orders respectively on 2^κ, then we say that Q_1 is *Borel-reducible* to Q_2 if there exists a κ-Borel map $f\colon 2^\kappa \to 2^\kappa$ such that for all $\eta, \xi \in 2^\kappa$ we have $\eta Q_1 \xi \iff f(\eta) Q_2 f(\xi)$ and this is also denoted by $Q_1 \leq_B Q_2$.

A quasi-order is Σ_1^1-*complete*, if it is $\Sigma_1^1(\kappa)$ and every $\Sigma_1^1(\kappa)$ quasi-order is Borel-reducible to it.

Theorem 1.10 ([5], Thm 3.1)**.** *($V = L$) The quasi-order $\sqsubseteq_\mu$ is Σ_1^1–complete, for every regular $\mu < \kappa$.*

To prove Theorem 1.10 we need to make some preparations before we start with the proof.

Definition 1.11.

- *Let us define a class function $F_\diamond\colon On \to L$. For all α, $F_\diamond(\alpha)$ is a pair (X_α, C_α) where $X_\alpha, C_\alpha \subseteq \alpha$, if α is a limit ordinal, then C_α is either a club or the empty set, and $C_\alpha = \emptyset$ when α is not a limit ordinal. We let $F_\diamond(\alpha) = (X_\alpha, C_\alpha)$ be the $<_L$-least pair such that for all $\beta \in C_\alpha$, $X_\beta \neq X_\alpha \cap \beta$ if α is a limit ordinal and such pair exists and otherwise we let $F_\diamond(\alpha) = (\emptyset, \emptyset)$.*
- *We let $C_\diamond \subseteq On$ be the class of all limit ordinals α such that for all $\beta < \alpha$, $F_\diamond \restriction \beta \in L_\alpha$. Notice that for every regular cardinal α, $C_\diamond \cap \alpha$ is a club.*

Definition 1.12. *For a given regular cardinal α and a subset $A \subset \alpha$, we define the sequence $(X_\gamma, C_\gamma)_{\gamma \in A}$ to be $(F_\diamond(\gamma))_{\gamma \in A}$, and the sequence $(X_\gamma)_{\gamma \in A}$ to be the sequence of sets X_γ such that $F_\diamond(\gamma) = (X_\gamma, C_\gamma)$ for some C_γ.*

By ZF$^-$ we mean ZFC+($V = L$) without the power set axiom. By ZF$^\diamond$ we mean ZF$^-$ with the following axiom:

> "For all regular cardinals $\mu < \alpha$ if $(S_\gamma, D_\gamma)_{\gamma \in \alpha}$ is such that for all $\gamma < \alpha$, $F_\diamond(\gamma) = (S_\gamma, D_\gamma)$, then $(S_\gamma)_{\gamma \in S_\mu^\alpha}$ is a diamond sequence."

Lemma 1.13 ([5], Lemma 3.4)**.** *($V = L$) For any Σ_1-formula $\varphi(\eta, x)$ with parameter $x \in 2^\kappa$, a regular cardinal $\mu < \kappa$, the following are equivalent for all $\eta \in 2^\kappa$:*

- $\varphi(\eta, x)$
- *$S \backslash A$ is non-stationary, where $S = \{\alpha \in S_\mu^\kappa \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$ and*

  $$A = \{\alpha \in C_\diamond \cap \kappa \mid \exists \beta > \alpha (L_\beta \models ZF^\diamond \wedge \varphi(\eta \restriction \alpha, x \restriction \alpha) \wedge r(\alpha))\}$$

  where $r(\alpha)$ is the formula "α is a regular cardinal".

Now we sketch the proof of 1.10.

Proof of Theorem 1.10 (sketch). Suppose Q is a Σ_1^1 quasi-order on 2^κ.

There is a Σ_1-formula of set theory $\psi(\eta, \xi) = \psi(\eta, \xi, x) = \exists k \varphi(k, \eta, \xi, x) \vee \eta = \xi$ with $x \in 2^\kappa$, such that for all $\eta, \xi \in 2^\kappa$,

$$(\eta, \xi) \in Q \Leftrightarrow \psi(\eta, \xi),$$

we added $\eta = \xi$ to $\psi(\eta, \xi)$, to ensure that when we reflect $\psi(\eta \restriction \alpha, \xi \restriction \alpha)$ we get a reflexive relation. Let $r(\alpha)$ be the formula "α is a regular cardinal" and $\psi^Q(\kappa)$ be the sentence with parameter κ that asserts that $\psi(\eta, \xi)$ defines a quasi-order on 2^κ. For all $\eta \in 2^\kappa$ and $\alpha < \kappa$, let

$$T_{\eta,\alpha} = \{p \in 2^\alpha \mid \exists \beta > \alpha (L_\beta \models \text{ZF}^\diamond \wedge \psi(p, \eta \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha))\}.$$

Let $(X_\alpha)_{\alpha \in S_\mu^\kappa}$ be the diamond sequence of Definition 1.12, and for all $\alpha \in S_\mu^\kappa$, let $\mathcal{X}_\alpha$ be the characteristic function of X_α. Define $\mathcal{F}\colon 2^\kappa \to 2^\kappa$ by

$$\mathcal{F}(\eta)(\alpha) = \begin{cases} 1 & \text{if } \mathcal{X}_\alpha \in T_{\eta,\alpha} \text{ and } \alpha \in S_\mu^\kappa \\ 0 & \text{otherwise} \end{cases}$$

Claim 1.14. *$\mathcal{F}$ is a reduction of Q into $\sqsubseteq_\mu$.*

□

Third Session

As it was sketch above, the combinatorial properties of L are essential for the reduction shown on Theorem 1.10. There are three different variations of Lemma 1.13, each variation is used to define a Borel reduction.

Lemma 1.15 ([2]). *($V = L$) For any Σ_1-formula $\varphi(\eta, x)$ with parameter $x \in 2^\kappa$, the following are equivalent for all $\eta \in 2^\kappa$:*

- $\varphi(\eta, x)$
- $A = \{\alpha < \kappa \mid \exists \beta > \alpha(L_\beta \models \mathrm{ZF}^- \wedge \varphi(\eta \restriction \alpha, x \restriction \alpha) \wedge r(\alpha))\}$ *contains a club, where $r(\alpha)$ is the formula "α is a regular cardinal".*

This variation was the one used in [2] to prove Theorem 1.15.

Lemma 1.16 ([3]). *($V = L$) For any Σ_1-formula $\varphi(\eta, x)$ with parameter $x \in 2^\kappa$, a regular cardinal $\mu < \kappa$, and a stationary set $S \subset S^\kappa_\mu$, the following are equivalent for all $\eta \in 2^\kappa$:*

- $\varphi(\eta, x)$
- *$S \backslash A$ is non-stationary, where*

$$A = \{\alpha \in S \mid \exists \beta > \alpha(L_\beta \models \varphi(\eta \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge s(\alpha))\}$$

where $r(\alpha)$ is the formula "α is a regular cardinal", and $s(\alpha)$ states that $S \cap \alpha$ is stationary and $S \cap \alpha \subset S^\alpha_\mu$ in the sense that we required β to be large enough to witness that every element of $S \cap \alpha$ has cofinality μ.

This Lemma was the one used in [3] to show that the relations $E^\kappa_{\mu\text{-club}}$ is $\Sigma^1_1(\kappa)$-complete under the assumption $V = L$. This is different from Lemma 1.15 because of the stationary set S. At the same time, in Lemma 1.16 is different from 1.13. In Lemma 1.16 S is fixed from the beginning and it is independent from η and in Lemma 1.13 S depends on η and the diamond sequence. This is the reason why Lemma 1.16 cannot be used to prove Theorem 1.10, and the reason to use $\mathrm{ZF}^\diamond$ and $C_\diamond$ to fix a diamond sequence $(X_\gamma)_{\gamma \in S^\kappa_\mu}$.

Proof of Theorem 1.13. Let $\mu < \kappa$ be a regular cardinal. Suppose that $\eta \in 2^\kappa$ is such that $\varphi(\eta, x)$ holds. Let θ be a cardinal large enough such that

$$L_\theta \models \mathrm{ZF}^\diamond \wedge \varphi(\eta, x) \wedge r(\kappa).$$

For each $\alpha < \kappa$, let

$$H(\alpha) = \mathrm{Sk}(\alpha \cup \{\kappa, \eta, x\})^{L_\theta}$$

and $\bar{H}(\alpha)$ the Mostowski collapse of $H(\alpha)$. Let

$$D = \{\alpha < \kappa \mid H(\alpha) \cap \kappa = \alpha\}.$$

Then D is a club set and $D \cap C_\diamond$ is a club. Since $H(\alpha)$ is an elementary submodel of L_θ and the Mostowski collapse $\bar{H}(\alpha)$ is equal to L_β for some $\beta > \alpha$, we have $D \cap C_\diamond \subseteq A$.

Suppose $\eta \in 2^\kappa$ is such that $\varphi(\eta, x)$ does not hold. Let $\mu < \kappa$ be a regular cardinal. Let θ be a large enough cardinal such that

$$L_\theta \models \mathrm{ZF}^\diamond \wedge \neg\varphi(\eta, x) \wedge r(\kappa).$$

Let C be an unbounded set which is closed under μ-limits (a μ-club). Let

$$H(\alpha) = \mathrm{Sk}(\alpha \cup \{\kappa, C, \eta, x, (X_\gamma, C_\gamma)_{\gamma \in S^\kappa_\mu}\})^{L_\theta}.$$

Let

$$D = \{\alpha \in S^\kappa_\mu \mid H(\alpha) \cap \kappa = \alpha\}$$

Notice that since $H(\alpha)$ is an elementary substructure of L_θ, then $H(\alpha)$ calculates all cofinalities correctly below α. Then D is an unbounded set, closed under μ-limits. Let $S = \{\alpha \in S^\kappa_\mu \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$ and α_0 be the least ordinal in $(\lim_\mu D) \cap S$ (where $\lim_\mu D$ is the set of ordinals of D that are μ-cofinal limits of elements of D). Since $\alpha_0 \in \lim_\mu D$, $\alpha_0 > \mu$. By the elementarity of each $H(\alpha)$ we conclude that $\alpha_0 \in C$.

Let $\bar{\beta}$ be such that $L_{\bar{\beta}}$ is equal to the Mostowski collapse of $H(\alpha_0)$. We will show that $\alpha_0 \notin A$. Suppose, towards a contradiction, that $\alpha_0 \in A$, thus $\alpha_0 \in C_\diamond \cap \kappa$. There exists $\beta > \alpha_0$ such that

$$L_\beta \models \mathrm{ZF}^\diamond \wedge \varphi(\eta \restriction \alpha_0, x \restriction \alpha_0) \wedge r(\alpha_0).$$

Since $\varphi(\eta, x)$ is a Σ_1-formula, β is a limit ordinal greater than $\bar{\beta}$.

Claim 1.17. *L_β satisfies the following:*

1. *For all $\gamma \in S \cap \alpha_0$, γ has cofinality μ.*

2. *$S \cap \alpha_0$ is a stationary subset of α_0.*

3. *$D \cap \alpha_0$ is a μ-club subset of a_0.*

Proof. 1. $H(\alpha_0)$ calculates all cofinalities correctly below α_0. Thus $L_{\bar{\beta}}$ calculates all cofinalities correctly below α_0. Since β is greater than $\bar{\beta}$, L_β calculates all cofinalities correctly below α_0. Since $S \cap \alpha_0 \subseteq S^\kappa_\mu$ in L, then $S \cap \alpha_0 \subseteq S^\kappa_\mu$ holds in L_β.

2. Since $\alpha_0 \in C_\diamond \cap \kappa$ and L_β satisfies ZF$^\diamond$ and $r(\alpha_0)$, L_β satisfies that $S \cap \alpha_0$ is a stationary subset of α_0.

3. Let $\alpha < \alpha_0$ be such that $L_\beta \models cf(\alpha) = \mu \ \wedge\ \bigcup(D \cap \alpha) = \alpha$, we will show that $L_\beta \models \alpha \in D \cap \alpha_0$. Since L_β calculates all cofinalities correctly below α_0, $L \models cf(\alpha) = \mu \wedge \bigcup(D \cap \alpha) = \alpha$. D is a μ-club in L, thus $L \models \alpha \in D$. Since $\alpha < \alpha_0$, $L \models \alpha \in D \cap \alpha_0$. We will finish the proof by showing that $L \models \alpha \in D \cap \alpha_0$ implies $L_\beta \models \alpha \in D \cap \alpha_0$.

   Notice that $H(\alpha_0)$ is a definable subset of L_θ and D is a definable subset of L_θ. By elementarity, $D \cap \alpha_0$ is a definable subset of $H(\alpha_0)$, we conclude that $D \cap \alpha_0$ is a definable subset of $L_{\bar{\beta}}$ and $D \cap \alpha_0 \in L_\beta$. Therefore $L_\beta \models \alpha \in D \cap \alpha_0$.

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□

Fourth Session

Let us continue with the proof of Theorem 1.10.

Proof.

Claim 1.18. *If $\eta\ Q\ \xi$, then $T_{\eta,\alpha} \subseteq T_{\xi,\alpha}$ for club-many α's.*

Proof. Suppose $\psi(\eta, \xi, x) = \exists k \varphi(k, \eta, \xi, x)$ holds and let k witnesses that. Let θ be a cardinal large enough such that $L_\theta \models \text{ZF}^\diamond \wedge \varphi(k, \eta, \xi, x) \wedge r(\alpha)$. For all $\alpha < \kappa$ let $H(\alpha) = \text{Sk}(\alpha \cup \{\kappa, k, \eta, \xi, x\})^{L_\theta}$. The set $D = \{\alpha < \kappa \mid H(\alpha) \cap \kappa = \alpha \wedge H(\alpha) \models \psi^Q(\alpha)\}$ is a club. Using the Mostowski collapse we have that

$$D' = \{\alpha < \kappa \mid \exists \beta > \alpha (L_\beta \models \text{ZF}^\diamond \wedge \varphi(k \restriction \alpha, \eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha))\}$$

contains a club. For all $\alpha \in D'$ and $p \in T_{\eta,\alpha}$ we have that

$$\exists \beta_1 > \alpha (L_{\beta_1} \models \text{ZF}^\diamond \wedge \psi(p, \eta \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha))$$

and

$$\exists \beta_2 > \alpha (L_{\beta_2} \models \text{ZF}^\diamond \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha)).$$

Therefore, for $\beta = \max\{\beta_1, \beta_2\}$ we have that

$$L_\beta \models \text{ZF}^\diamond \wedge \psi(p, \eta \restriction \alpha, x \restriction \alpha) \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha).$$

Since $\psi^Q(\alpha)$ holds and so transitivity holds for $\psi(\eta, \xi)$ in L_β, we conclude that

$$L_\beta \models \text{ZF}^\diamond \wedge \psi(p, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha)$$

so $p \in T_{\xi,\alpha}$ and $T_{\eta,\alpha} \subseteq T_{\xi,\alpha}$. This holds for all $\alpha \in D'$. □

By the previous claim, we conclude that if $\eta\ Q\ \xi$, then there is a μ-club C such that for every $\alpha \in C$ it holds that $\mathcal{X}_\alpha \in T_{\eta,\alpha} \Rightarrow \mathcal{X}_\alpha \in T_{\xi,\alpha}$. Therefore $(\mathcal{F}(\eta)^{-1}\{1\} \backslash \mathcal{F}(\xi)^{-1}\{1\}) \cap C = \emptyset$, and $\mathcal{F}(\eta)\ \sqsubseteq_\mu\ \mathcal{F}(\xi)$.

For the other direction, suppose $\neg\psi(\eta, \xi, x)$ holds. Let $S = \{\alpha \in S^\kappa_\mu \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$. Since $(X_\gamma)_{\gamma \in S^\kappa_\mu}$ is a diamond sequence, S is a stationary set. By Lemma 1.13 we know that $S \backslash A$ is stationary, where

$$A = \{\alpha \in C_\diamond \cap \kappa \mid \exists \beta > \alpha (L_\beta \models \text{ZF}^\diamond \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha))\}.$$

Since for all $\alpha \in S \backslash A$ we have that $X_\alpha = \eta^{-1}\{1\} \cap \alpha$, so $\mathcal{X}_\alpha \in T_{\eta,\alpha}$. We conclude that for all $\alpha \in S \backslash A$, $\mathcal{F}(\eta)(\alpha) = 1$. On the other hand, for all $\alpha \in S \backslash A$ it holds that

$$\forall \beta > \alpha (L_\beta \not\models \text{ZF}^\diamond \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha))$$

so

$$\forall\beta>\alpha(L_\beta\not\models \mathrm{ZF}^\diamond\wedge\psi(\mathcal{X}_\alpha,\xi\restriction\alpha,x\restriction\alpha)\wedge r(\alpha)).$$

Therefore

$$\forall\beta>\alpha(L_\beta\not\models \mathrm{ZF}^\diamond\wedge\psi(\mathcal{X}_\alpha,\xi\restriction\alpha,x\restriction\alpha)\wedge r(\alpha)\wedge\psi^Q(\alpha))$$

we conclude that $\mathcal{X}_\alpha\notin T_{\xi,\alpha}$, and $\mathcal{F}(\xi)(\alpha)=0$. Hence, for all $\alpha\in S\backslash A$, $\mathcal{F}(\eta)(\alpha)=1$ and $\mathcal{F}(\xi)(\alpha)=0$. Since $S\backslash A$ is stationary, we conclude that $\mathcal{F}(\eta)^{-1}\{1\}\backslash\mathcal{F}(\xi)^{-1}\{1\}$ is stationary and $\mathcal{F}(\eta)\not\subseteq_\mu\mathcal{F}(\xi)$.

□

Theorem 1.19 ([2], Thm 18)**.** *($V=L$) κ-Borel* $=\Sigma_1^1(\kappa)$.*

Proof. Because of Example 1.1 and the previous Lemma, it is enough to prove the following Claim.

Claim 1.20. *Assume $f\colon 2^\kappa\to 2^\kappa$ is a κ-Borel function and $B\subset 2^\kappa$ is κ-Borel*. Then $f^{-1}[B]$ is κ-Borel*.*

Let (T_B,H_B) be a κ-Borel*-code for B. Define the κ-Borel*-code (T_A,H_A) by letting $T_B=T_A$ and $H_A(b)=f^{-1}[H_B(b)]$ for every branch b of T_B. Let A be the κ-Borel*-set coded by (T_A,H_A). Clearly, $\mathbf{II}\uparrow B^*(T_B,H_B,\eta)$ if and only if $\mathbf{II}\uparrow B^*(T_A,H_A,f^{-1}(\eta))$, so $f^{-1}[B]=A$. □

Theorem 1.21 ([10], Corollary 34)**.** *Suppose A and B are disjoint $\Sigma_1^1(\kappa)$ sets. There are κ-Borel* sets C_0 and C_1 such that $A\subseteq C_0$, $B\subseteq C_1$, and C_0 and C_1 are duals.*

Theorem 1.22 ([2], Theorem 17)**.** *$\Delta_1^1(\kappa)\subseteq\kappa$-Borel**

Lemma 1.23 ([4], Corollary 3.2)**.** *It is consistently that $\Delta_1^1(\kappa)\subsetneq\kappa$-Borel* $\subsetneq\Sigma_1^1(\kappa)$.*

Definition 1.24. *Fix a bijection $\pi\colon\kappa^{<\omega}\to\kappa$. For every $\eta\in\kappa^\kappa$ define the $\mathcal{L}$-structure $\mathcal{A}_\eta$ with domain κ as follows: For every relation P_m with arity n, every tuple $(a_1,a_2,\dots,a_n)$ in κ^n satisfies*

$$(a_1,a_2,\dots,a_n)\in P_m^{\mathcal{A}_\eta}\iff\eta(\pi(m,a_1,a_2,\dots,a_n))\geq 0.$$

Theorem 1.25 ([2], Theorem 18)**.**

1. *κ-Borel$\subsetneq\Delta_1^1(\kappa)$*
2. *$\Delta_1^1(\kappa)\subsetneq\Sigma_1^1(\kappa)$*

Proof. 1. Let $\xi\mapsto(T_\xi,h_\xi)$ be a continuous coding of the κ-Borel*-codes with T a $\kappa^+\omega$-tree, such that for all $\kappa^+\omega$-tree, T, and h, there is ξ such that $T_\xi,h_\xi=(T,h)$.

Claim 1.26. *The set $B=\{(\eta,\xi)\mid\eta$ is in the set coded by $(T_\xi,h_\xi)\}$ is $\Sigma_1^1(\kappa)$ and is not κ-Borel, otherwise $D=\{\eta\mid(\eta,\eta)\notin B\}$ would be κ-Borel (Hint: use the set $C=\{(\eta,\xi,\sigma)\mid\sigma$ is a winning strategy for $\mathbf{II}$ in $B^*(T_\xi,h_\xi,\eta)\}$).*

2.

Claim 1.27. *There is $A\subseteq 2^\kappa\times 2^\kappa$ such that if $B\subseteq 2^\kappa$ is a $\Sigma_1^1(\kappa)$ set, then there is $\eta\in 2^\kappa$ such that $B=\{\xi\mid(\xi,\eta)\in A\}$ (Hint: the construction used in the classical case works too).*

The set $D=\{\eta\mid(\eta,\eta)\in A\}$ is $\Sigma_1^1(\kappa)$ but not $\Pi_1^1(\kappa)$.

□

Question 1.28. *Is it consistent that $\Delta_1^1(\kappa)=\kappa$-Borel*?*

Definition 1.29 (The isomorphism relation)**.** *Assume T is a complete first order theory in a countable vocabulary. We define $\cong_T$ as the relation*

$$\{(\eta,\xi)\in\kappa^\kappa\times\kappa^\kappa\mid(\mathcal{A}_\eta\models T,\mathcal{A}_\xi\models T,\mathcal{A}_\eta\cong\mathcal{A}_\xi)\text{ or }(\mathcal{A}_\eta\not\models T,\mathcal{A}_\xi\not\models T)\}.$$

Theorem 1.30 ([2], Theorem 70)**.** *If T is a classifiable theory, then $\cong_T$ is $\Delta_1^1(\kappa)$.*

Theorem 1.31 ([2], Theorem 87)**.** *Suppose that for all $\gamma<\kappa$, $\gamma^\omega<\kappa$ and T is a stable unsuperstable countable theory. Then $E^2_{\omega\text{-club}}\leq_c\cong^\kappa_T$.* □

Theorem 1.32 ([2], Theorem 79)**.** *Suppose that $\kappa=\lambda^+=2^\lambda$ and $\lambda^{<\lambda}=\lambda$.*

1. *If T is unstable or superstable with OTOP, then $E^2_{\lambda\text{-club}}\leq_c\cong^\kappa_T$.*
2. *If $\lambda\geq 2^\omega$ and T is superstable with DOP, then $E^2_{\lambda\text{-club}}\leq_c\cong^\kappa_T$.* □

Corollary 1.33. *($V=L$) Suppose that κ is the successor of a regular uncountable cardinal. If T is a non-classifiable countable theory, then $\cong_T$ is a Σ_1^1-complete relation.*

2 The Main Gap in the generalized Borel-reducibility hierarchy

Fifth Session

Shelah's Main Gap Theorem states the following.

Theorem 2.1 ([11] Main Gap Theorem)**.** *For every T first order complete theory over a countable vocabulary. Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α. One of the following holds:*

1. *If T is shallow superstable without DOP and without OTOP, then $\forall\alpha > 0$ $I(T,\aleph_\alpha) < \beth_{\omega_1}(|\alpha|)$.*
2. *If T is not superstable, or superstable and deep or with DOP or with OTOP, then for every uncountable cardinal α, $I(T,\alpha) = 2^\alpha$.*

This gives us a notion of complexity, a theory is more complex if it has more models. Unfortunately, the main gap also tells us that with this notion of complexity a theory T is either too complex, for every uncountable cardinal α $I(T,\alpha) = 2^\alpha$, or it is not so complex, i.e. $\forall\alpha > 0$ $I(T,\aleph_\alpha) < \beth_{\omega_1}(|\alpha|)$. The aim of study the Main Gap in the generalized Borel reducibility hierarchy is to obtain a more refined complexity notion in which different theories have different complexities, and satisfies a counterpart of the Main Gap theorem:

If T_1 and T_2 are first order complete theories over a countable vocabulary such that T_1 satisfies the first item of the Main Gap and T_2 satisfies the second item of the Main Gap theorem, then T_1 is less complex than T_2.

With the notions explained in the previous session, we can define the desire complexity notion:

T_1 is as much as complex as T_2 if and only $\cong_{T_1}\leq_B\cong_{T_2}$.

To study this notion of complexity for first order complete theories over countable vocabularies, we will divide the theories in two classes (as the Main Gap suggested), classifiable and non-classifiable theories. The only difference is that we will not require a theory to be shallow in order to be classifiable. Some authors require shallow for classifiable theories, we will see why in our case it make sense to not require it.

Definition 2.2.

- *A first order complete theory over a countable vocabulary, T, is classifiable if it is superstable without DOP and without OTOP.*
- *A first order complete theory over a countable vocabulary, T, is non-classifiable if it satisfies one of the following:*
  1. *T is stable unsuperstable;*
  2. *T is superstable with DOP;*
  3. *T is superstable with OTOP;*
  4. *T is unstable.*

In previous sessions we saw that it is consistently true that if T_1 is classifiable and T_2 is not classifiable then $\cong_{T_1}\leq_B\cong_{T_2}$, this is a consequence of the diamond principle. In particular in L all non-classifiable theories are Σ_1^1-complete [5]. During the following lectures we will focus on the results of ZFC.

Classifiable Theories

Let us start with the case of classifiable theories. The following is the usual Ehrenfeucht-Fraïssé game but coded in a particular way for our purposes.

Definition 2.3. *(Ehrenfeucht-Fraïssé game) Fix $\{X_\gamma\}_{\gamma<\kappa}$ an enumeration of the elements of $\mathcal{P}_\kappa(\kappa)$ and $\{f_\gamma\}_{\gamma<\kappa}$ an enumeration of all the functions with domain in $\mathcal{P}_\kappa(\kappa)$ and range in $\mathcal{P}_\kappa(\kappa)$. For every pair of structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ and $\alpha<\kappa$, the $EF^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ is a game played by the players* **I** *and* **II** *as follows.*
In the n-th move, first **I** *choose an ordinal $\beta_n<\alpha$ such that $X_{\beta_n}\subset\alpha$, $X_{\beta_{n-1}}\subseteq X_{\beta_n}$, and then* **II** *an ordinal $\theta_n<\alpha$ such that $dom(f_{\theta_n}), rang(f_{\theta_n})\subset\alpha$, $X_{\beta_n}\subseteq dom(f_{\theta_n})\cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}}\subseteq f_{\theta_n}$ (if $n=0$ then $X_{\beta_{n-1}}=\emptyset$ and $f_{\theta_{n-1}}=\emptyset$). The game finishes after ω moves. The player* **II** *wins if $\cup_{i<\omega}f_{\theta_i}: A\restriction_\alpha\to B\restriction_\alpha$ is a partial isomorphism, otherwise the player* **I** *wins.*

We write **I** $\uparrow$ $\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ if **I** has a winning strategy in the game $\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$. We write **II** $\uparrow$ $\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ if **II** has a winning strategy.

Lemma 2.4 ([6], Lemma 2.4)**.** *If $\mathcal{A}$ and $\mathcal{B}$ are structures with domain κ, then the following hold:*

- **II** $\uparrow$ *$EF^\kappa_\omega(\mathcal{A}\restriction\kappa,\mathcal{B}\restriction\kappa)\iff$* **II** $\uparrow$ *$EF^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α.*

- $\mathbf{I}\uparrow EF^\kappa_\omega(\mathcal{A}\restriction\kappa,\mathcal{B}\restriction\kappa)\iff\mathbf{I}\uparrow EF^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ *for club-many* α.

Proof. It is easy to see that if $\sigma:\kappa^{<\omega}\to\kappa$ is a winning strategy for **II** in the game $\text{EF}^\kappa_\omega(\mathcal{A}\restriction\kappa,\mathcal{B}\restriction\kappa)$, then $\sigma\restriction\alpha^{<\alpha}$ is a winning strategy for **II** in the game $\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ if $\sigma[\alpha^{<\alpha}]\subseteq\alpha$. So $\mathbf{II}\uparrow\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for α a closed point of σ.

We conclude that if $\mathbf{II}\uparrow\text{EF}^\kappa_\omega(\mathcal{A}\restriction\kappa,\mathcal{B}\restriction\kappa)$, then $\mathbf{II}\uparrow\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α. The same holds for **I**. To show the other direction, notice that $\text{EF}^\kappa_\omega(\mathcal{A}\restriction\kappa,\mathcal{B}\restriction\kappa)$ is a determined game, so if **II** doesn't have a winning strategy, then **I** has a winning strategy. Therefore, if **II** doesn't have a winning strategy in the game $\text{EF}^\kappa_\omega(\mathcal{A}\restriction\kappa,\mathcal{B}\restriction\kappa)$, then $\mathbf{I}\uparrow\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α, and **II** cannot have a winning strategy in $\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α. □

The reason to introduce these games is that we can characterize classifiable theories with these games.

Theorem 2.5 ([11], XIII Theorem 1.4)**.** *If T is a classifiable theory, then every two models of T that are $L_{\infty,\kappa}$-equivalent are isomorphic.*

Theorem 2.6 ([2], Theorem 10)**.** *$L_{\infty,\kappa}$-equivalence is equivalent to EF^κ_ω-equivalence.*

From these two theorems we know that if T is a classifiable theory, then for any $\mathcal{A}$ and $\mathcal{B}$ models of T with domain κ,

$$\mathbf{II}\uparrow\text{EF}^\kappa_\omega(\mathcal{A},\mathcal{B})\iff\mathcal{A}\cong\mathcal{B}$$

$$\mathbf{I}\uparrow\text{EF}^\kappa_\omega(\mathcal{A},\mathcal{B})\iff\mathcal{A}\not\cong\mathcal{B}.$$

From the previous Lemma we know the following two hold for any $\mathcal{A}$ and $\mathcal{B}$ models of a classifiable theory (with domain κ):

- $\mathcal{A}\cong\mathcal{B}\iff\mathbf{II}\uparrow\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α.
- $\mathcal{A}\not\cong\mathcal{B}\iff\mathbf{I}\uparrow\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α.

Theorem 2.7 ([6], Theorem 2.8)**.** *Assume T is a classifiable theory and $\mu<\kappa$ a regular cardinal, then*

$$\cong_T\ \leq_c\ E^\kappa_{\mu\text{-}club}.$$

Proof. Define the relation $F_\alpha\subseteq\kappa^\kappa\times\kappa^\kappa$ by $(\eta,\xi)\in F_\alpha$ if $\mathbf{II}\uparrow\text{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$. Define the reduction $\mathcal{F}:\kappa^\kappa\to\kappa^\kappa$ as follows,

$$\mathcal{F}(\eta)(\alpha)=\begin{cases}f_\eta(\alpha) & \text{if } cf(\alpha)=\mu, \mathcal{A}_\eta\restriction_\alpha\models T \text{ and } F_\alpha \text{ is an equivalence relation}\\ 0 & \text{in other case}\end{cases}$$

where $f_\eta(\alpha)$ is a code in $\kappa\backslash\{0\}$ for the F_α equivalence class of $\mathcal{A}_\eta\restriction_\alpha$.

Claim 2.8. *There are club many α's such that F_α is an equivalence relation.*

Proof. It follows by intersecting the clubs generated by the closed points of winning strategies. □

The previous observation finishes the proof. □

We conclude this section with a couple of observations. As a corollary of Theorem 1.21, $A\subseteq\kappa^\kappa$ is a $\Delta^1_1(\kappa)$ set if there is a κ-Borel*-code (T,h), such that for all $\eta\in\kappa^\kappa$ the game $B^*(T,h,\eta)$ is determined. By Theorems 2.4 and 2.6 we conclude that if T is classifiable, then $\cong_T$ is a $\Delta^1_1(\kappa)$ equivalence relation (this is the prove of Theorem 1.30). This is not the case for some non-classifiable theories.

Theorem 2.9 ([2], Theorem 71)**.** *If T is unstable, or superstable with OTOP, or superstable with DOP and $\kappa>\omega_1$, then $\cong_T$ is not a $\Delta^1_1(\kappa)$ equivalence relation.*

Friedman, Hyttinen, and Kulikov proved in [2] the consistency of "$\cong_T$ is not $\Delta^1_1(\kappa)$ for every stable unsuperstable theory T".

Question 2.10. *Is it consistently true that there is a stable unsuperstable theory T such that $\cong_T$ is not a $\Delta^1_1(\kappa)$ equivalence relation?*

Finally, when T is classifiable and shallow the result is stronger, as the Main Gap suggest.

Theorem 2.11 ([2], Theorem 68)**.** *If $\kappa>2^\omega$, and T is classifiable and shallow, then $\cong_T$ is a κ-Borel equivalence relation*

Coloured Trees

To study the non-classifiable theories we need to introduce the coloured trees. Coloured trees are very useful to reduce $E^{\kappa}_{\mu\text{-club}}$ or $E^{2}_{\mu\text{-club}}$ to $\cong_T$, for certain μ and nonclassifiable theory T (see [2], [3], [6], [9]). In [2] and [3] the coloured trees used had height $\omega+2$ and were used to study the case when κ is a successor cardinal. In [6] the coloured trees had height $\omega+2$ and were used to study the case when κ is an inaccessible cardinal. In these lectures we will use the coloured trees of [9], i.e. trees of uncountable height and κ inaccessible. Given a tree t, for every $x\in t$ we denote the order type of $\{y\in t|y<x\}$. Let us define $t_\alpha=\{x\in t|ht(x)=\alpha\}$ and $t_{<\alpha}=\cup_{\beta<\alpha}t_\beta$, and denote by $x\restriction\alpha$ the unique $y\in t$ such that $y\in t_\alpha$ and $y\le x$. If $x,y\in t$ and $\{z\in t|z<x\}=\{z\in t|z<y\}$, then we say that x and y are $\sim$-related, $x\sim y$, and we denote by $[x]$ the equivalence class of x for $\sim$. An α,β-tree is a tree t with the following properties:

- $|[x]|<\alpha$ for every $x\in t$.
- All the branches have order type less than β in t.
- t has a unique root.
- If $x,y\in t$, x and y has no immediate predecessors and $x\sim y$, then $x=y$.

Definition 2.12. *Let λ be an uncountable cardinal. A coloured tree is a pair (t,c), where t is a κ^+, $(\lambda+2)$-tree and c is a map $c:t_\lambda\to\kappa\backslash\{0\}$.*

Definition 2.13. *Let (t,c) be a coloured tree, suppose $(I_\alpha)_{\alpha<\kappa}$ is a collection of subsets of t that satisfies:*

- *for each $\alpha<\kappa$, I_α is a downward closed subset of t.*
- $\bigcup_{\alpha<\kappa}I_\alpha=t$.
- *if $\alpha<\beta<\kappa$, then $I_\alpha\subset I_\beta$.*
- *if γ is a limit ordinal, then $I_\gamma=\bigcup_{\alpha<\gamma}I_\alpha$.*
- *for each $\alpha<\kappa$ the cardinality of I_α is less than κ.*

We call $(I_\alpha)_{\alpha<\kappa}$ a filtration of t.

Definition 2.14. *Let t be a coloured tree and $\mathcal{I}=(I_\alpha)_{\alpha<\kappa}$ a filtration of t. Define $H_{\mathcal{I},t}\in\kappa^\kappa$ as follows. Fix $\alpha<\kappa$. Let B_α be the set of all $x\in t_\lambda$ that are not in I_α, but $x\restriction\theta\in I_\alpha$ for all $\theta<\lambda$.*

- *If B_α is non-empty and there is β such that for all $x\in B_\alpha$, $c(x)=\beta$, then let $H_{\mathcal{I},t}(\alpha)=\beta$*
- *Otherwise let $H_{\mathcal{I},t}(\alpha)=0$*

We will call a filtration good if for every α, $B_\alpha\neq\emptyset$ implies that c is constant on B_α.

Lemma 2.15 ([9]). *Suppose (t_0,c_0) and (t_1,c_1) are isomorphic coloured trees, and $\mathcal{I}=(I_\alpha)_{\alpha<\kappa}$ and $\mathcal{J}=(J_\alpha)_{\alpha<\kappa}$ are good filtrations of (t_0,c_0) and (t_1,c_1) respectively. Then $H_{\mathcal{I},t_0}\ E^{\kappa}_{\lambda\text{-club}}\ H_{\mathcal{J},t_1}$*

Proof. Let $F:(t_0,c_0)\to(t_1,c_1)$ be a coloured tree isomorphism. Define $F\mathcal{I}=(F[I_\alpha])_{\alpha<\kappa}$. It is easy to see that $F[I_\alpha]$ is a downward closed subset of t_1. Clearly $F[I_\alpha]\subset F[I_\beta]$ when $\alpha<\beta$ and for γ a limit ordinal, $\cup_{\alpha<\gamma}F[I_\alpha]=F[I_\gamma]$. If $x\in t_1$ then there exists $y\in t_0$ and $\alpha<\kappa$ such that $F(y)=x$ and $y\in I_\alpha$, therefore $x\in F[I_\alpha]$ and $\cup_{\alpha<\kappa}F[I_\alpha]=t_1$. Since F is an isomorphism, $|F[I_\alpha]|=|I_\alpha|<\kappa$ for every $\alpha<\kappa$. So $F\mathcal{I}$ is a filtration of t_1.
For every α, $B^{\mathcal{I}}_\alpha\neq\emptyset$ implies that $B^{F\mathcal{I}}_\alpha\neq\emptyset$. On the other hand, $\mathcal{I}$ is a good filtration, then when $B^{\mathcal{I}}_\alpha\neq\emptyset$, c_0 is constant on $B^{\mathcal{I}}_\alpha$. Since F is colour preserving, c_1 is constant on $B^{F\mathcal{I}}_\alpha$, we conclude that $F\mathcal{I}$ is a good filtration and $H_{\mathcal{I},t_0}(\alpha)=H_{F\mathcal{I},t_1}(\alpha)$.
Notice that $F[I_\alpha]=J_\alpha$ implies $H_{\mathcal{I},t_0}(\alpha)=H_{\mathcal{J},t_1}(\alpha)$. Therefore it is enough to show that $C=\{\alpha|F[I_\alpha]=J_\alpha\}$ is an λ-club. By the definition of a filtration, for every sequence $(\alpha_i)_{i<\theta}$ in C, cofinal to γ, $J_\gamma=\bigcup_{i<\theta}J_{\alpha_i}=\bigcup_{i<\theta}F[I_{\alpha_i}]=F[I_\gamma]$, so C is closed. To show that C is unbounded, choose $\alpha<\kappa$. Define the succession $(\alpha_i)_{i<\lambda}$ by induction. For $i=0$, $\alpha_0=\alpha$. For every limit ordinal γ, when n is odd let $\alpha_{\gamma+n+1}$ be the least ordinal bigger than $\alpha_{\gamma+n}$ such that $F[I_{\alpha_{\gamma+n}}]\subset J_{\alpha_{\gamma+n+1}}$ (such ordinal exists because κ is regular, and $\mathcal{J}$ and $F\mathcal{I}$ are filtrations, specially $|F[I_{\alpha_{\gamma+n}}]|<\kappa$). For every limit ordinal γ, when n is even let $\alpha_{\gamma+n+1}$ be the least ordinal bigger than $\alpha_{\gamma+n}$ such that $J_{\alpha_{\gamma+n}}\subset F[I_{\alpha_{\gamma+n+1}}]$ (such ordinal exists because κ is regular, and $\mathcal{J}$ and $F\mathcal{I}$ are filtrations, specially $|J_{\alpha_n}|<\kappa$). Define $\alpha_\gamma=\bigcup_{i<\gamma}\alpha_i$, then $J_{\alpha_\gamma}=\bigcup_{i<\gamma}J_{\alpha_i}=\bigcup_{i<\gamma}F[I_{\alpha_i}]=F[I_{\alpha_\gamma}]$. Clearly $\bigcup_{i<\lambda}J_{\alpha_i}=\bigcup_{i<\lambda}F[I_{\alpha_i}]$ and $\cup_{i<\lambda}\alpha_i\in C$. □

Order the set $\lambda\times\kappa\times\kappa\times\kappa\times\kappa$ lexicographically, $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) > (\beta_1,\beta_2,\beta_3,\beta_4,\beta_5)$ if for some $1 \leq k \leq 5$, $\alpha_k > \beta_k$ and for every $i < k$, $\alpha_i = \beta_i$. Order the set $(\lambda \times \kappa \times \kappa \times \kappa \times \kappa)^{\leq\lambda}$ as a tree by inclusion.
Define the tree (I_f, d_f) as, I_f the set of all strictly increasing functions from some $\theta \leq \lambda$ to κ and for each η with domain λ, $d_f(\eta) = f(sup(rang(\eta)))$.
For every pair of ordinals α and β, $\alpha < \beta < \kappa$ and $i < \lambda$ define

$$R(\alpha,\beta,i) = \bigcup_{i<j\leq\lambda} \{\eta : [i,j) \to [\alpha,\beta) | \eta \text{ strictly increasing}\}.$$

Definition 2.16. *Assume κ is an inaccessible cardinal. If $\alpha < \beta < \kappa$ and $\alpha,\beta,\gamma \neq 0$, let $\{P^{\alpha,\beta}_\gamma | \gamma < \kappa\}$ be an enumeration of all downward closed subtrees of $R(\alpha,\beta,i)$ for all i, in such a way that each possible coloured tree appears cofinally often in the enumeration. And the tree $P^{0,0}_0$ is (I_f, d_f).*

This enumeration is possible because κ is inaccessible; there are at most $|\bigcup_{i<\lambda} \mathcal{P}(R(\alpha,\beta,i))| \leq \lambda \times \kappa = \kappa$ downward closed coloured subtrees, and at most $\kappa \times \kappa^{<\kappa} = \kappa$ coloured trees. Denote by $Q(P^{\alpha,\beta}_\gamma)$ the unique ordinal number i such that $P^{\alpha,\beta}_\gamma \subset R(\alpha,\beta,i)$.

Definition 2.17. *Assume κ is an inaccessible cardinal. Define for each $f \in \kappa^\kappa$ the coloured tree (J_f, c_f) by the following construction.*
For every $f \in \kappa^\kappa$ define $J_f = (J_f, c_f)$ as the tree of all $\eta : s \to \lambda \times \kappa^4$, where $s \leq \lambda$, ordered by extension, and such that the following conditions hold for all $i, j < s$:
Denote by η_i, $1 \leq i \leq 5$, the functions from s to κ that satisfies, $\eta(n) = (\eta_1(n), \eta_2(n), \eta_3(n), \eta_4(n), \eta_5(n))$.

1. *$\eta \restriction n \in J_f$ for all $n < s$.*
2. *η is strictly increasing with respect to the lexicographical order on $\lambda \times \kappa^4$.*
3. *$\eta_1(i) \leq \eta_1(i+1) \leq \eta_1(i) + 1$.*
4. *$\eta_1(i) = 0$ implies $\eta_2(i) = \eta_3(i) = \eta_4(i) = 0$.*
5. *$\eta_2(i) \geq \eta_3(i)$ implies $\eta_2(i) = 0$.*
6. *$\eta_1(i) < \eta_1(i+1)$ implies $\eta_2(i+1) \geq \eta_3(i) + \eta_4(i)$.*
7. *For every limit ordinal α, $\eta_k(\alpha) = sup_{\beta<\alpha}\{\eta_k(\beta)\}$ for $k \in \{1,2\}$.*
8. *$\eta_1(i) = \eta_1(j)$ implies $\eta_k(i) = \eta_k(j)$ for $k \in \{2,3,4\}$.*
9. *If for some $k < \lambda$, $[i,j) = \eta_1^{-1}\{k\}$, then*
$$\eta_5 \restriction [i,j) \in P^{\eta_2(i),\eta_3(i)}_{\eta_4(i)}.$$
Note that 7 implies $Q(P^{\eta_2(i),\eta_3(i)}_{\eta_4(i)}) = i$.
10. *If $s = \lambda$, then either*
    (a) *there exists an ordinal number m such that for every $k < m$ $\eta_1(k) < \eta_1(m)$, for every $k' \geq m$ $\eta_1(k) = \eta_1(m)$, and the color of η is determined by $P^{\eta_2(m),\eta_3(m)}_{\eta_4(m)}$:*
    $$c_f(\eta) = c(\eta_5 \restriction [m,\lambda))$$
    where c is the colouring function of $P^{\eta_2(m),\eta_3(m)}_{\eta_4(m)}$.

    or

    (b) *there is no such ordinal m and then $c_f(\eta) = f(sup(rang(\eta_5)))$.*

Lemma 2.18 ([9]). *Assume κ is an inaccessible cardinal, then for every $f, g \in \kappa^\kappa$ the following holds*

$$f\ E^\kappa_{\lambda\text{-club}}\ g \Leftrightarrow J_f \cong J_g$$

Proof. By Lemma 2.4, it is enough to prove the following properties of J_f

1. There is a good filtration $\mathcal{I}$ of J_f, such that $H_{\mathcal{I},J_f}\ E^\kappa_{\lambda\text{-club}}\ f$.

2. If $f\ E^\kappa_{\lambda\text{-club}}\ g$, then $J_f \cong J_g$.

Notice that for any $k \in rang(\eta_1)$ if $\eta_5 \restriction [i,j) \in P^{\eta_2(i),\eta_3(i)}_{\eta_4(i)}$, when $[i,j) = \eta_1^{-1}(k)$ and if $i+1<j$, then $\eta_5 \restriction [i,j)$ is strictly increasing. If $\eta_1(i) < \eta_1(i+1)$, by Definition 2.6 item 6, $\eta_2(i+1) \geq \eta_3(i) + \eta_4(i)$, so $\eta_5(i) < \eta_3(i) \leq \eta_2(i+1) \leq \eta_5(i+1)$. If α is a limit ordinal, by Definition 2.6 items 7 and 8, $\eta_5(\beta) < \eta_2(\beta+1) < \eta_2(\alpha) \leq \eta_5(\alpha)$ it holds for every $\beta < \alpha$. Thus η_5 is strictly increasing. If $\eta \restriction n \in J_f$ for every n, then $\eta \in J_f$. Clearly every maximal branch has order type $\lambda+1$, every chain $\eta \restriction 1 \subset \eta \restriction 2 \subset \eta \restriction 3 \subseteq \cdots$ of any length, has a unique limit in the tree, and every element in t_θ, $\theta < \lambda$, has an infinite number of successors (at most κ), therefore $J_f \in CT^\lambda_*$. For each $\alpha < \kappa$ define J^α_f as

$$J^\alpha_f = \{\eta \in J_f | rang(\eta) \subset \lambda \times (\beta)^4 \text{ for some } \beta < \alpha\}.$$

Suppose $rang(\eta_1) = \lambda$. As it was mentioned before, η_5 is increasing and $sup(rang(\eta_3)) \geq sup(rang(\eta_5)) \geq sup(rang(\eta_2))$. By Definition 2.6 item 6 $sup(rang(\eta_2)) \geq sup(rang(\eta_3))$ and $sup(rang(\eta_2)) \geq sup(rang(\eta_4))$, this lead us to

$$sup(rang(\eta_4)) \leq sup(rang(\eta_3)) = sup(rang(\eta_5)) = sup(rang(\eta_2)). \tag{1}$$

When $\eta \restriction k \in J^\alpha_f$ holds for every $k \in \lambda$, it can be concluded that $sup(rang(\eta_5)) \leq \alpha$, if in addition $\eta \notin J^\alpha_f$, then

$$sup(rang(\eta_5)) = \alpha. \tag{2}$$

Claim 2.19. *Suppose $\xi \in J^\alpha_f$ and $\eta \in J_f$. If $dom(\xi)$ a successor ordinal smaller than λ, $\xi \subsetneq \eta$ and for every k in $dom(\eta)\backslash dom(\xi)$, $\eta_1(k) = \xi_1(max(dom(\xi)))$ and $\eta_1(k) > 0$, then $\eta \in J^\alpha_f$.*

Proof. Assume $\xi, \eta \in J_f$ are as in the assumption. Let $\beta_i = \xi_i(max(dom(\xi)))$, for $i \in \{2,3,4\}$. Since $\xi \in J^\alpha_f$, then there exists $\beta < \alpha$ such that $\beta_2, \beta_3, \beta_4 < \beta$. By Definition 2.6 item 8 for every $k \in dom(\eta)\backslash dom(\xi)$, $\eta_i(k) = \beta_i$ for $i \in \{2,3,4\}$. Therefore, by Definition 2.6 item 9 and the definition of $P^{\beta_2,\beta_3}_{\beta_4}$, we conclude $\eta_5(k) < \beta_3 < \beta$, so $\eta \in J^\alpha_f$. □

Claim 2.20. $|J_f| = \kappa$, *$\mathcal{J} = (J^\alpha_f)_{\alpha<\kappa}$ is a good filtration of J_f and $H_{\mathcal{J},J_f}\ E^\kappa_{\lambda\text{-club}}\ f$*

Proof. Clearly $J_f = \cup_{\alpha<\kappa} J^\alpha_f$, J^α_f is a downward closed subset of J_f, and $J^\alpha_f \subset J^\beta_f$ when $\alpha < \beta$. Since κ is inaccessible, we conclude $|J^\alpha_f| < \kappa$ and $|J_f| = \kappa$. Finally, when γ is a limit ordinal

$$\begin{aligned} J^\gamma_f &= \{\eta \in J_f | \exists \beta < \gamma (rang(\eta) \subset \omega \times (\beta)^4)\} \\ &= \{\eta \in J_f | \exists \alpha < \gamma, \exists \beta < \alpha (rang(\eta) \subset \omega \times (\beta)^4)\} \\ &= \textstyle\bigcup_{\alpha<\gamma} J^\alpha_f \end{aligned}$$

Suppose α has cofinality λ, and $\eta \in J_f\backslash J^\alpha_f$ satisfies $\eta \restriction k \in J^\alpha_f$ for every $k < \lambda$. By the previous claim, η satisfies Definition 2.6 item 10 (a) only if $\eta_1(n) = 0$ for every $n \in \lambda$. So η_1, η_2, η_3 and η_4 are constant zero, and $c_f(\eta) = d_f(\eta_5)$, where d_f is the colouring function of $P^{0,0}_0 = I_f$, $c_f(\eta) = f(sup(rang(\eta_5)))$. When η satisfies Definition 2.6 item 10 (b), $c_f(\eta) = f(sup(rang(\eta_5)))$.
In both cases, $c_f(\eta) = f(\alpha)$. Therefore, if $B_\alpha \neq \emptyset$ then c_f is constant on B_α and $\mathcal{J}$ is a good filtration.
By Definition 2.3 and since $\mathcal{J}$ is a good filtration, $H_{\mathcal{J},J_f}(\alpha) = f(\alpha)$. □

Claim 2.21. *If $f\ E^\kappa_{\lambda\text{-club}}\ g$, then $J_f \cong J_g$.*

Proof. Let $C' \subseteq \{\alpha < \kappa | f(\alpha) = g(\alpha)\}$ be an λ-club testifying $f\ E^\kappa_{\lambda\text{-club}}\ g$, and let $C \supset C'$ be the closure of C' under limits. By induction we are going to construct an isomorphism between J_f and J_g.
We define continuous increasing sequences $(\alpha_i)_{i<\kappa}$ of ordinals and $(F_{\alpha_i})_{i<\kappa}$ of partial color-preserving isomorphism from J_f to J_g such that:

a) If i is a successor, then α_i is a successor ordinal and there exists $\beta \in C$ such that $\alpha_{i-1} < \beta < \alpha_i$ and thus if i is a limit, $\alpha_i \in C$.

b) Suppose that $i = \gamma + n$, where γ is a limit ordinal or 0, and $n < \omega$ is even. Then $dom(F_{\alpha_i}) = J^{\alpha_i}_f$.

c) Suppose that $i = \gamma + n$, where γ is a limit ordinal or 0, and $n < \omega$ is odd. Then $rang(F_{\alpha_i}) = J^{\alpha_i}_g$.

d) If $dom(\xi) < \lambda$, $\xi \in dom(F_{\alpha_i})$, $\eta \restriction dom(\xi) = \xi$ and for every $k \geq dom(\xi)$

$$\eta_1(k) = \xi_1(sup(dom(\xi))) \text{ and } \eta_1(k) > 0$$

then $\eta \in dom(F_{\alpha_i})$. Similar for $rang(F_{\alpha_i})$.

e) If $\xi \in dom(F_{\alpha_i})$ and $k < dom(\xi)$, then $\xi \restriction k \in dom(F_{\alpha_i})$.

f) For all $\eta \in dom(F_{\alpha_i})$, $dom(\eta) = dom(F_{\alpha_i}(\eta))$.

For every ordinal α denote by $M(\alpha)$ the ordinal that is order isomorphic to the lexicographic order of $\lambda \times \alpha^4$.

First step (i=0).
Let $\alpha_0 = \beta + 1$ for some $\beta \in C$. Let γ be an ordinal such that there is a coloured tree isomorphism $h : P_\gamma^{0,M(\beta)} \to J_f^{\alpha_0}$ and $Q(P_\gamma^{0,M(\beta)}) = 0$. It is easy to see that such γ exists, by the way our enumeration was chosen.
Since $P_\gamma^{0,M(\beta)}$ and $J_f^{\alpha_0}$ are closed under initial segments, then $|dom(h^{-1}(\eta))| = |dom(\eta)|$. Also both domains are intervals containing zero, therefore $dom(h^{-1}(\eta)) = dom(\eta)$.
Define $F_{\alpha_0}(\eta)$ for $\eta \in J_f^{\alpha_0}$ as follows, let $F_{\alpha_0}(\eta)$ be the function ξ with $dom(\xi) = dom(\eta)$, and for all $\kappa < dom(\xi)$

- $\xi_1(k) = 1$
- $\xi_2(k) = 0$
- $\xi_3(k) = M(\beta)$
- $\xi_4(k) = \gamma$
- $\xi_5(k) = h^{-1}(\eta)(k)$

To check that $\xi \in J_g$, we will check every item of Definition 2.6. Since $rang(F_{\alpha_0}) = \{1\} \times \{0\} \times \{M(\beta)\} \times \{\gamma\} \times P_\gamma^{0,M(\beta)}$, ξ satisfies 1. Also $\xi_5 = h^{-1}(\eta) \in P_\gamma^{0,M(\beta)}$, by definition of $P_\gamma^{\alpha,\beta}$, we now that ξ_5 is strictly increasing with respect to the lexicographic order, then ξ satisfies item 2. Notice that ξ is constant in every component except for ξ_5, therefore ξ satisfies the items 3, 6, 7, 8, 10 (a). Clearly $\xi_1(i) \neq 0$, so ξ satisfies item 4. Since $\xi_2(k) = 0$ for every k, then ξ satisfies 5. Notice that $[0, \lambda) = \xi_1^{-1}(1)$ but $P_{\xi_4(k)}^{\xi_2(k),\xi_3(k)} = P_\gamma^{0,M(\beta)}$ for every k, therefore $\xi_5 \in P_{\xi_4(0)}^{\xi_2(0),\xi_3(0)}$ and ξ satisfies 7.
Let us show that the conditions a)-f) are satisfied, the conditions a) and c) are clearly satisfied. By the way F_{α_0} was defined, $dom(F_{\alpha_0}) = J_f^{\alpha_0}$ and $dom(\eta) = dom(F_{\alpha_0}(\eta))$, these are the conditions b), e) and f). Since $dom(F_{\alpha_0}) = J_f^{\alpha_0}$, the Claim 2.7.1 implies d) for $dom(F_{\alpha_0})$. For d) with $rang(F_{\alpha_0})$, suppose $\xi \in rang(F_{\alpha_0})$ and $\eta \in J_g$ are as in the assumption. Then $\eta_1(k) = \xi_1(k) = 1$ for every $k < dom(\eta)$, by 8 in J_g we have that $\eta_2(k) = \xi_2(k) = 0$, $\eta_3(k) = \xi_3(k) = M(\beta)$ and $\eta_4(k) = \xi_4(k) = \gamma$ for every $k < dom(\eta)$. By 9 in J_g, $\eta_5 \in P_\gamma^{0,M(\beta)}$ and since $rang(F_{\alpha_0}) = \{1\} \times \{0\} \times \{M(\beta)\} \times \{\gamma\} \times P_\gamma^{0,M(\beta)}$, we can conclude that $\eta \in rang(F_{\alpha_0})$.

Odd successor step.
Suppose that $j < k$ is a successor ordinal such that $j = \beta_j + n_j$ for some limit ordinal (or 0) β_j and an odd integer n_j. Assume α_l and F_{α_l} are defined for every $l < j$ satisfying the conditions a)-f).
Let $\alpha_j = \beta + 1$ where $\beta \in C$ is such that $\beta > \alpha_{j-1}$ and $rang(F_{\alpha_{j-1}}) \subset J_g^\beta$, such a β exists because $|rang(F_{\alpha_{j-1}})| \leq 2^{|\alpha_{j-1}|}$ and κ is strongly inaccessible.
When $\eta \in rang(F_{\alpha_{j-1}})$ has domain $m < \lambda$, define

$$W(\eta) = \{\zeta | dom(\zeta) = [m, s), m < s \leq \lambda, \eta^\frown \langle m, \zeta(m) \rangle \notin rang(F_{\alpha_{j-1}}) \text{ and } \eta^\frown \zeta \in J_g^{\alpha_j}\}$$

with the color function $c_{W(\eta)}(\zeta) = c_g(\eta^\frown \zeta)$ for every $\zeta \in W(\eta)$ with $s = \lambda$. Denote $\xi' = F_{\alpha_{j-1}}^{-1}(\eta)$, $\alpha = \xi'_3(m-1) + \xi'_4(m-1)$ (if m is a limit ordinal, then $\alpha = sup_{\theta < m} \xi_2(\theta)$) and $\theta = \alpha + M(\alpha_j)$. Now choose an ordinal γ_η such that $Q(P_{\gamma_\eta}^{\alpha,\theta}) = m$ and there is an isomorphism $h_\eta : P_{\gamma_\eta}^{\alpha,\theta} \to W(\eta)$. We will define F_{α_j} by defining its inverse such that $rang(F_{\alpha_j}) = J_g^{\alpha_j}$.
Each $\eta \in J_g^{\alpha_j}$ satisfies one of the followings:

(*) $\eta \in rang(F_{\alpha_{j-1}})$.

(**) $\exists m < dom(\eta)(\eta \restriction m \in rang(F_{\alpha_{j-1}}) \wedge \eta \restriction (m+1) \notin rang(F_{\alpha_{j-1}}))$.

(***) $\forall m < dom(\eta)(\eta \restriction (m+1) \in rang(F_{\alpha_{j-1}}) \wedge \eta \notin rang(F_{\alpha_{j-1}}))$.

We define $\xi = F_{\alpha_j}^{-1}(\eta)$ as follows. There are three cases:

Case η satisfies $(*)$.
Define $\xi(n) = F_{\alpha_{j-1}}^{-1}(\eta)(n)$ for all $n < dom(\eta)$.

Case η satisfies $(**)$.
This case is divided in two subcases, when m is limit ordinal and when m is successor ordinal. Let m witnesses (**) for η and suppose m is a successor ordinal. For every $n < dom(\xi)$

- If $n < m$, then $\xi(n) = F^{-1}_{\alpha_{j-1}}(\eta \restriction m)(n)$.
- For every $n \geq m$. Let
  - $\xi_1(n) = \xi_1(m-1) + 1$
  - $\xi_2(n) = \xi_3(m-1) + \xi_4(m-1)$
  - $\xi_3(n) = \xi_2(m) + M(\alpha_j)$
  - $\xi_4(n) = \gamma_{\eta\restriction m}$
  - $\xi_5(n) = h^{-1}_{\eta\restriction m}(\eta \restriction [m, dom(\eta)))(n)$

Note that, $\eta \restriction [m, dom(\eta))$ is an element of $W(\eta \restriction m)$, this makes possible the definition of ξ_5.
Let us check the items of Definition 2.6 to see that $\xi \in J_f$. Clearly item 1 is satisfied. By induction hypothesis, $\xi \restriction m$ is increasing, $\xi_1(m) = \xi_1(m-1) + 1$ so $\xi(m-1) < \xi(m)$, and ξ_k is constant on $[m, \lambda)$ for $k \in \{1, 2, 3, 4\}$, since $h^{-1}_{\eta\restriction m}(\eta) \in P^{\alpha,\theta}_{\gamma_\eta}$, then ξ_5 is increasing, and we conclude that ξ is increasing with respect to the lexicographic order, so ξ satisfies item 2. Also we conclude $\xi_1(i) \leq \xi_1(i+1) \leq \xi_1(i) + 1$, so ξ satisfies item 3. For every $i < \lambda$, $\xi_1(i) = 0$ implies $i < m$, so $\xi(i) = F^{-1}_{\alpha_{j-1}}(\eta \restriction m)(i)$ and by the induction hypothesis ξ satisfies item 4. By the induction hypothesis, $\xi \restriction m \in J_f$, since $\xi_2(n) = \xi_3(m-1) + \xi_4(m-1)$ holds for every $n \geq m$, we conclude that ξ satisfies 5. By the induction hypothesis, for every $i + 1 < m$, $\xi_1(i) < \xi_1(i+1)$ implies $\xi_2(i+1) \geq \xi_3(i) + \xi_4(i)$, on the other hand $\xi_1(i) = \xi_1(j)$ implies $\xi_k(i) = \xi_k(j)$ for $k \in \{2, 3, 4\}$, clearly $\xi_2(m) \geq \xi_3(m-1) + \xi_4(m-1)$ and $\xi_k(i) = \xi_k(i+1)$ for $i \geq m$ and $k \in \{2, 3, 4\}$, then ξ satisfies items 6 and 8.
By the induction hypothesis, $\xi \restriction m \in J_f$, since $\xi_1(n) = \xi_1(m-1) + 1$ and $\xi_2(n) = \xi_3(m-1) + \xi_4(m-1)$ hold for every $n \geq m$, we conclude that ξ satisfies 7. Suppose $[i, j) = \xi_1^{-1}(k)$ for some k in $rang(\xi)$. Either $j < m$ or $m = i$. If $j < m$, by the induction hypothesis $\xi_5 \restriction [i, j) \in P^{\xi_2(i),\xi_3(i)}_{\xi_4(i)}$, if $[i, j) = [m, dom(\xi))$, then $\xi_5 \restriction [i, j) = h^{-1}_{\eta\restriction m}(\eta \restriction [m, dom(\xi))) \in P^{\xi_2(m),\xi_3(m)}_{\xi_4(m)}$, ξ thus satisfies item 9. Since ξ is constant on $[m, \lambda)$, ξ satisfies 10 (a). Finally by item 10 (a) when $dom(\zeta) = \lambda$, $c_f(\xi) = c(\xi_5 \restriction [m, \lambda))$, where c is the color of $P^{\xi_2(m),\xi_3(m)}_{\xi_4(m)}$. Since $\xi_5 \restriction [m, \lambda) = h^{-1}_{\eta\restriction m}(\eta \restriction [m, \lambda))$, $c_f(\xi) = c(h^{-1}_{\eta\restriction m}(\eta \restriction [m, \lambda)))$ and since h is an isomorphism, $c_f(\xi) = c_{W(\eta\restriction m)}(\eta \restriction [m, \lambda)) = c_g(\eta)$.

Let m witnesses (**) for η and suppose m is a limit ordinal. For every $n < dom(\xi)$

- If $n < m$, then $\xi(n) = F^{-1}_{\alpha_{j-1}}(\eta \restriction m)(n)$.
- For every $n \geq m$. Let
  - $\xi_1(n) = sup_{\theta<m}\xi_1(\theta)$
  - $\xi_2(n) = sup_{\theta<m}\xi_2(\theta)$
  - $\xi_3(n) = \xi_2(m) + M(\alpha_j)$
  - $\xi_4(n) = \gamma_{\eta\restriction m}$
  - $\xi_5(n) = h^{-1}_{\eta\restriction m}(\eta \restriction [m, dom(\eta)))(n)$

Note that, $\eta \restriction [m, dom(\eta))$ is an element of $W(\eta \restriction m)$, this makes possible the definition of ξ_5.
Let us check the items of Definition 2.6 to see that $\xi \in J_f$. Clearly item 1 is satisfied. By induction hypothesis, $\xi \restriction m$ is increasing, $\xi_1(m) = sup_{\theta<m}\xi_1(\theta)$ so $\xi(\theta) < \xi(m)$ for every $\theta < m$, and ξ_k is constant on $[m, \lambda)$ for $k \in \{1, 2, 3, 4\}$, since $h^{-1}_{\eta\restriction m}(\eta) \in P^{\alpha,\theta}_{\gamma_\eta}$, then ξ_5 is increasing, and we conclude that ξ is increasing with respect to the lexicographic order, so ξ satisfies item 2. Also we conclude $\xi_1(i) \leq \xi_1(i+1) \leq \xi_1(i) + 1$, so ξ satisfies item 3. For every $i < \lambda$, $\xi_1(i) = 0$ implies $i < m$, so $\xi(i) = F^{-1}_{\alpha_{j-1}}(\eta \restriction m)(i)$ and by the induction hypothesis ξ satisfies item 4. By the induction hypothesis, $\xi \restriction m \in J_f$, since $\xi_2(n) = sup_{\theta<m}\xi_2(\theta)$ holds for every $n \geq m$, we conclude that ξ satisfies 5. By the induction hypothesis, for every $i + 1 < m$, $\xi_1(i) < \xi_1(i+1)$ implies $\xi_2(i+1) \geq \xi_3(i) + \xi_4(i)$, on the other hand $\xi_1(i) = \xi_1(j)$ implies $\xi_k(i) = \xi_k(j)$ for $k \in \{2, 3, 4\}$, clearly $\xi_2(m) \geq sup_{\theta<m}\xi_3(\theta)$ and $\xi_k(i) = \xi_k(j)$ for $j, i \geq m$ and $k \in \{2, 3, 4\}$, then ξ satisfies items 6 and 8.
By the induction hypothesis, $\xi \restriction m \in J_f$, since $\xi_1(n) = sup_{\theta<m}\xi_1(\theta)$ and $\xi_2(n) = sup_{\theta<m}\xi_2(\theta)$ hold for every $n \geq m$, we conclude that ξ satisfies 7. Suppose $[i, j) = \xi_1^{-1}(k)$ for some k in $rang(\xi)$. Either $j < m$ or $m = i$, notice that if $i < m < j$, then $\eta \restriction (m+1) \in rang(F_{\alpha_{j-1}})$). If $j < m$, by the induction hypothesis $\xi_5 \restriction [i, j) \in P^{\xi_2(i),\xi_3(i)}_{\xi_4(i)}$, if $[i, j) = [m, dom(\xi))$, then $\xi_5 \restriction [i, j) = h^{-1}_{\eta\restriction m}(\eta \restriction [m, dom(\xi))) \in P^{\xi_2(m),\xi_3(m)}_{\xi_4(m)}$, ξ thus satisfies item 9. Since ξ is constant on $[m, \lambda)$, ξ satisfies 10 (a). Finally by item 10 (a) when $dom(\zeta) = \lambda$, $c_f(\xi) = c(\xi_5 \restriction [m, \lambda))$, where c is the color of $P^{\xi_2(m),\xi_3(m)}_{\xi_4(m)}$. Since $\xi_5 \restriction [m, \lambda) = h^{-1}_{\eta\restriction m}(\eta \restriction [m, \lambda))$, $c_f(\xi) = c(h^{-1}_{\eta\restriction m}(\eta \restriction [m, \lambda)))$ and since h is an isomorphism, $c_f(\xi) = c_{W(\eta\restriction m)}(\eta \restriction [m, \lambda)) = c_g(\eta)$.

Case η satisfies $(***)$.

Clearly $dom(\eta)=\lambda$, by the induction hypothesis and condition d), $rang(\eta)=\lambda$, otherwise $\eta\in rang(F_{\alpha_{j-1}})$. Let $F^{-1}_{\alpha_j}(\eta)=\xi=\cup_{n<\lambda}F^{-1}_{\alpha_{j-1}}(\eta\restriction n)$, by the induction hypothesis, ξ is well defined. Since for every $n<\lambda$, $\xi\restriction n\in J_f$, then $\xi\in J_f$. Let us check that $c_f(\xi)=c_g(\eta)$. First note that $\xi\notin J_f^{\alpha_{j-1}}$, otherwise by the induction hypothesis f),

$$F_{\alpha_{j-1}}(\xi)=\bigcup_{n<\lambda}F_{\alpha_{j-1}}(\xi\restriction n)=\bigcup_{n<\lambda}\eta\restriction n=\eta$$

giving us $\eta\in rang(F_{\alpha_{j-1}})$. By the equation (2), $sup(rang(\xi_5))=\alpha_{j-1}$ and ξ satisfies item 10 b) in J_f, therefore $c_f(\xi)=f(\alpha_{j-1})$. Also by the definition of J_f^α and since $\xi\restriction n\in J_f^{\alpha_{j-1}}$ for every $n<\lambda$, α_{j-1} is a limit ordinal and by condition a), $j-1$ is a limit ordinal and $\alpha_{j-1}\in C$. The conditions b) and c) ensure $rang(F_{\alpha_{j-1}})=J_f^{\alpha_{j-1}}$. This implies, $\eta\notin J_f^{\alpha_{j-1}}$. By the equation (2), $sup(rang(\eta_5))=\alpha_{j-1}$. Therefore α_{j-1} has cofinality λ, $\alpha_{j-1}\in C'$ and $f(\alpha_{j-1})=g(\alpha_{j-1})$. By item 10 b) in J_g, $c_g(\eta)=g(\alpha_{j-1})=f(\alpha_{j-1})=c_f(\xi)$.

Next we show that F_{α_i} is a color preserving partial isomorphism. We already showed that F_{α_i} preserve the colors, so we only need to show that

$$\eta\subsetneq\xi\Leftrightarrow F^{-1}_{\alpha_i}(\eta)\subsetneq F^{-1}_{\alpha_i}(\xi). \tag{3}$$

From left to right.

When $\eta,\xi\in rang(F_{\alpha_{i-1}})$, the induction hypothesis implies (3) from left to right. If $\eta\in rang(F_{\alpha_{i-1}})$ and $\xi\notin rang(F_{\alpha_{i-1}})$, the construction implies (3) from left to right. Let us assume $\eta,\xi\notin rang(F_{\alpha_{i-1}})$, then η,ξ satisfy (**). Let m_1 and m_2 be the respective ordinal numbers that witness (**) for η and ξ, respectively. Notice that $m_2<dom(\eta)$, otherwise, $\eta\in rang(F_{\alpha_{i-1}})$. If $m_1<m_2$, clearly $\eta\in rang(F_{\alpha_{i-1}})$ what is not the case. A similar argument shows that $m_2<m_1$ cannot hold. We conclude that $m_1=m_2$ and by the construction of F_{α_i}, $F^{-1}_{\alpha_i}(\eta)\subsetneq F^{-1}_{\alpha_i}(\xi)$.

From right to left.

When $\eta,\xi\in rang(F_{\alpha_{i-1}})$, the induction hypothesis implies (3) from right to left. If $\eta\in rang(F_{\alpha_{i-1}})$ and $\xi\notin rang(F_{\alpha_{i-1}})$, the construction implies (3) from right to left. Let us assume $\eta,\xi\notin rang(F_{\alpha_{i-1}})$, then η,ξ satisfy (**). Let m_1 and m_2 be the respective ordinal numbers that witness (**) for η and ξ, respectively. Notice that $m_2<dom(\eta)$, otherwise, $F^{-1}_{\alpha_i}(\eta)=F^{-1}_{\alpha_{i-1}}(\eta)$ and $\eta\in rang(F_{\alpha_{i-1}})$. Let us denote by θ the inverse map $F^{-1}_{\alpha_i}$ (e.g. $\theta(\zeta)=F^{-1}_{\alpha_i}(\zeta)$), and the first component by θ_1 (e.g. $\theta_1(\zeta)=F^{-1}_{\alpha_i}(\zeta)_1$).
If $m_1<m_2$ and m_2 is a successor ordinal, then

$$\begin{array}{rcl}\theta_1(\eta)(m_2-1) & = & (\theta(\xi)\restriction_{m_2})_1(m_2-1)\\ & < & \theta_1(\xi\restriction_{m_2})(m_2-1)+1\\ & = & \theta_1(\eta)(m_2)\\ & = & \theta_1(\eta)(m_2-1).\end{array}$$

If $m_1<m_2$ and m_2 is a limit ordinal, then

$$\begin{array}{rcl}\forall\gamma\in[m_1,m_2)\ \ \theta_1(\eta)(\gamma) & = & (\theta(\xi)\restriction_{m_2})_1(\gamma)\\ & < & sup_{n<m_2}\theta_1(\xi\restriction_{m_2})(n)\\ & = & \theta_1(\eta)(m_2)\\ & = & \theta_1(\eta)(\gamma).\end{array}$$

This cannot hold. A similar argument shows that $m_2<m_1$ cannot hold. We conclude that $m_1=m_2$.
By the induction hypothesis $F^{-1}_{\alpha_{i-1}}(\eta\restriction m_1)=F^{-1}_{\alpha_{i-1}}(\xi\restriction m_2)$ implies $\eta\restriction m_1=\xi\restriction m_2$ (also implies $h_{\eta\restriction m_1}=h_{\xi\restriction m_2}$). Since $F^{-1}_{\alpha_{i-1}}(\eta\restriction m_1)(n)=F^{-1}_{\alpha_i}(\eta)(n)$ for all $n<m_1$, we only need to prove that $\eta\restriction[m_1,dom(\eta))\subsetneq\xi\restriction[m_2,dom(\xi))$. But $h_{\eta\restriction m_1}$ is an isomorphism and $F^{-1}_{\alpha_i}(\eta)_5(n)=F^{-1}_{\alpha_i}(\xi)_5(n)$ for every $n\geq m_1$, so $h^{-1}_{\eta\restriction m_1}(\eta\restriction[m_1,dom(\eta)))(n)=h^{-1}_{\xi\restriction m_2}(\xi\restriction[m_2,dom(\xi)))(n)$. Therefore $\eta\restriction[m_1,dom(\eta))\subsetneq\xi\restriction[m_2,dom(\xi))$.

Let us check that this three constructions satisfy the conditions a)-f).
When i is a successor we have $\alpha_{i-1}<\beta<\alpha_i=\beta+1$ for some $\beta\in C$, this is the condition a). Clearly the three cases satisfy b). We defined $F^{-1}_{\alpha_i}$ according to (*), (**), or (***); since every $\eta\in J_g^{\alpha_j}$ satisfies one of these, we conclude $rang(F_{\alpha_i})=J_g^{\alpha_j}$ which is the condition c).
Let us show that the F_{α_i} satisfy condition d). Let ξ and η be as in the assumptions of condition d) for domain. Notice that if $\xi\in dom(F_{\alpha_{i-1}})$ then the induction hypothesis ensure that $\eta\in dom(F_{\alpha_i})$. Suppose $\xi\notin dom(F_{\alpha_{i-1}})$, then $F_{\alpha_i}(\xi)\notin rang(F_{\alpha_{i-1}})$. Since $dom(\xi)<\lambda$, so $F_{\alpha_i}(\xi)$ satisfies (**). Let m be the number witnessing it. If m is a limit ordinal, then $dom(\xi)\geq m+1$, therefore $\xi\restriction m+1\in J_f^{\alpha_i}$ and by Claim 2.7.1 $\eta\in J_f^{\alpha_i}$. If m is a successor ordinal, then $\xi\in J_f^{\alpha_i}$ and by Claim 2.7.1 $\eta\in J_f^{\alpha_i}$. By item 8 in $J_f^{\alpha_i}$, η_k is constant on $[m,dom(\eta))$ for $k\in\{2,3,4\}$, now by Definition 2.6 item 9 in $J_f^{\alpha_i}$, $\eta_5\restriction[m,dom(\eta))\in P^{\alpha,\beta}_{\gamma_{\xi\restriction m}}$. Let $\zeta=h_{\xi\restriction m}(\eta_{[m,dom(\eta))})$, then $\eta=F^{-1}_{\alpha_i}(F_{\alpha_i}(\xi\restriction m)^\frown\zeta)$ and $\eta\in dom(F_{\alpha_i})$.
Using the same argument, the condition d) can be proved.

For the conditions e) and f), notice that ξ was constructed such that $dom(\xi) = dom(\eta)$ and $\xi \restriction k \in dom(F_{\alpha_i})$ which are these conditions.

Even successor step.
Suppose that $j < k$ is a successor ordinal such that $j = \beta_j + n_j$ for some limit ordinal (or 0) β_j and an even integer n_j. Assume α_l and F_{α_l} are defined for every $l < j$ satisfying conditions a)-f).
Let $\alpha_j = \beta + 1$ where $\beta \in C$ such that $\beta > \alpha_{j-1}$ and $dom(F_{\alpha_{j-1}}) \subset J_f^{\beta}$, such a β exists because $|dom(F_{\alpha_{j-1}})| \leq 2^{|\alpha_{j-1}|}$ and κ is strongly inaccessible. The construction of F_{α_j} such that $dom(F_{\alpha_j}) = J_f^{\alpha_i}$ follows as in the odd successor step, with the equivalent definitions for $dom(F_{\alpha_j})$ and $J_f^{\alpha_i}$. Notice that for every $\eta \in J_f^{\alpha_j}$, there are only the following cases:

(*) $\eta \in dom(F_{\alpha_{j-1}})$.

(**) $\exists m < dom(\eta)(\eta \restriction m \in dom(F_{\alpha_{j-1}}) \wedge \eta \restriction (m+1) \notin dom(F_{\alpha_{j-1}}))$.

Limit step.
Assume j is a limit ordinal. Let $\alpha_j = \cup_{i<j}\alpha_i$ and $F_{\alpha_j} = \cup_{i<j}F_{\alpha_i}$, clearly $F_{\alpha_j} : J_f^{\alpha_j} \to J_g$ and satisfies condition c). Since for i successor, α_i is the successor of an ordinal in C, then $\alpha_j \in C$ and satisfies the condition a). Also F_{α_j} is a partial isomorphism. Remember that $\cup_{i<j} J_f^{\alpha_i} = J_f^{\alpha_j}$, the same for J_g. By the induction hypothesis and the conditions b) and c) for $i < j$, we have $dom(F_{\alpha_j}) = J_f^{\alpha_j}$ (this is the condition b)) and $rang(F_{\alpha_j}) = J_g^{\alpha_j}$. This and Claim 2.7.1 ensure that condition d) is satisfied. By the induction hypothesis, for every $i < j$, F_{α_i} satisfies conditions e) and f), then F_{α_j} satisfies conditions e) and f). □

Define $F = \cup_{i<\kappa} F_{\alpha_i}$, clearly, it is an isomorphism between J_f and J_g. □

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