ON BOREL SETS IN IDEAL TOPOLOGIES

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Abstract. We study the Borel and analytic subsets of the spaces ${}^{\kappa}\kappa$ and ${}^{\kappa}2$ endowed with ideal topologies, where κ is a regular uncountable cardinal. We establish that the Borel hierarchy does not collapse in any ideal topology and prove that every Borel set in such a topology is analytic. In particular, when the ideal contains an unbounded set, the class of analytic sets coincides with the entire powerset. Furthermore, we show that the Approximation Lemma holds for ideal topologies.

Contents

1. Introduction

Where classical descriptive set theory examines the Borel and analytic hierarchies on Polish spaces like the Cantor space and Baire space, generalized descriptive set theory investigates analogous hierarchies on generalized versions of these spaces, obtained by replacing ω with an uncountable cardinal κ (or with its cofinality). The study of generalized descriptive set theory began with the foundational works of Vaught [Vau75] and Väänänen [Vä91], and the literature on ${}^{\kappa}2$ and ${}^{\kappa}\kappa$ for uncountable cardinals κ has since become extensive. In most works in this area, these spaces are equipped with the bounded topology under the assumption that $\kappa^{<\kappa} = \kappa$, yielding interesting results on their κ^+-Borel and κ-analytic subsets (see e.g. [MV93, FHK14, HK18, LS15, ACRP25]). The bounded topology is particularly natural due to its applications in model theory and infinitary logics [Vau75, MV93, SV00, She01, SV02, She04, DV11, FHK14, MMR21, Mor23].

However, meaningful variants of this classical framework have emerged in the literature. One approach is to drop or weaken the cardinal assumption: either working

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with regular κ while permitting $\kappa^{<\kappa} > \kappa$ [HMV25], or adopting the weaker assumption $2^{<\kappa} = \kappa$ and allowing κ to be singular [AMR22, DR25, ACRP25, MRP25]. Another direction involves changing the topology itself. While papers on infinitary logics predominantly use the bounded topology, works in general topology [Kur66, Ü82, Kra04, Ili12, CK13] typically employ either the product or box topology. Recent developments have shown that the product topology can be indispensable in certain contexts in generalized descriptive set theory, as demonstrated in [AMR22, Sections 12 and 13] and [DR25, Sections 5 and 6].

This brings us to the framework central to this article. In [HKSW22], Holy, Koelbing, Schlicht, and Wohofsky introduced the *ideal topology* $\tau_{\mathcal{I}}$ on ${}^{\kappa}2$ and ${}^{\kappa}\kappa$, where κ is a regular uncountable cardinal and $\mathcal{I}$ is an ideal extending the bounded ideal (see Definition 2.3). This topology refines the bounded topology and provides a unifying framework that encompasses both the bounded and product topologies as special cases. While [HKSW22] studied many properties of ideal topologies and initiated the investigation of $\mathcal{I}$-Borel sets, their analysis of the $\mathcal{I}$-Borel hierarchy remained incomplete, and they left open fundamental questions about its structure. Moreover, the notion of analytic sets was not addressed in the ideal topology context. Our work provides a comprehensive treatment of both topics, resolving the main open question from [HKSW22].

We now describe the content of the paper in detail. Section 2 introduces the ideal topology on ${}^{\kappa}2$ and ${}^{\kappa}\kappa$ and establishes its basic properties (Lemma 2.5). In Section 3, we characterize continuous functions on these spaces via monotone and domain-increasing functions (Definition 3.1).

Section 4 provides a comprehensive description of the κ^+-Borel hierarchy on spaces equipped with the ideal topology $\tau_{\mathcal{I}}$. We provide a rigorous treatment of all relevant definitions and key foundational results about $\mathcal{I}$-Borel sets. This includes natural features of the $\mathcal{I}$-Borel hierarchy, like the property of being increasing (Corollary 4.5), the closure properties of the pointclasses $\mathcal{I}$-$\mathbf{\Sigma}^0_\alpha$, $\mathcal{I}$-$\mathbf{\Pi}^0_\alpha$ and $\mathcal{I}$-$\mathbf{\Delta}^0_\alpha$ appearing in it (Proposition 4.11) and the length of the hierarchy. The authors of [HKSW22] identified the following as one of their most important open questions:

Question (Question 1, [HKSW22])**.** Do the $\mathcal{I}$-Borel sets form a strict hierarchy of length κ^+?

We answer this question affirmatively in Theorem 4.9, establishing that the $\mathcal{I}$-Borel hierarchy does not collapse for any ideal $\mathcal{I}$ extending the bounded ideal.

Section 5 introduces and analyzes the class of $\mathcal{I}$-analytic sets. Surprisingly, this class behaves quite differently from its classical counterpart. We prove in Theorem 5.2 that, when the ideal $\mathcal{I}$ contains an unbounded set, the class of $\mathcal{I}$-analytic sets coincides with the entire powerset of the space. This dramatic collapse occurs for all reasonable definitions of $\mathcal{I}$-analyticity (Corollary 5.3 and Proposition 5.5), revealing a fundamental distinction between the bounded topology and ideal topology settings.

Finally, Section 6 extends the approximation method to all ideal topologies. The Approximation Lemma is one of the principal tools for constructing continuous reductions in generalized descriptive set theory (see [FHK14, HM17, FMR21]), and our generalization (Lemma 6.6) makes this powerful technique available in the broader context of ideal topologies.

Throughout this paper, we work in ZFC and we assume that κ is an uncountable regular cardinal.

2. Ideals and ideal topologies

An ideal $\mathcal{I}$ on κ is **κ-complete** if for every $\mathcal{J} \subseteq \mathcal{I}$, $|\mathcal{J}| < \kappa$ implies $\bigcup \mathcal{J} \in \mathcal{I}$, and it is proper if $\mathcal{I} \neq \mathscr{P}(\kappa)$. Some well known κ-complete proper ideals are b, the collection of bounded subsets of κ, and NS_κ, the collection of nonstationary subsets of κ. A set $\mathcal{B} \subseteq \mathcal{I}$ is a **basis for the ideal** $\mathcal{I}$ if for every $D \in \mathcal{I}$ there exists $D' \in \mathcal{B}$ with $D \subseteq D'$. For example, $\{\alpha \mid \alpha < \kappa\}$ is a basis (of size κ) for the bounded ideal. Note that there are proper ideals other than the bounded ideal which have a basis of size κ, e.g., the ideal generated by the bounded ideal together with a set A, where A is an unbounded subset of κ with unbounded complement (see Example 3.4). Given $\mathcal{I}$ and $\mathcal{J}$ ideals on κ, we say that $\mathcal{I}$ is **$\mathcal{J}$-tall** if for all $A \subseteq \kappa$ such that $A \notin \mathcal{J}$, there is $B \in \mathcal{I} \setminus \mathcal{J}$ such that $B \subseteq A$. We say that $\mathcal{I}$ is **tall** if it is b-tall, and **stationary tall** if it is NS_κ-tall.

From this point onward (and unless stated otherwise), we assume that

$\mathcal{I}$ is a κ-complete proper ideal on κ that extends the ideal of bounded sets.

Definition 2.1. An ideal $\mathcal{I}$ has the **approximation property** if there is a strictly increasing continuous[1] sequence $\langle B_\alpha \mid \alpha < \kappa \rangle$ in $\mathcal{I}$ such that $B_0 = \emptyset$, and for all $B \in \mathcal{I}$ there is $\alpha < \kappa$ such that $B \subsetneq B_\alpha$. We call $\langle B_\alpha \mid \alpha < \kappa \rangle$ an approximation sequence of $\mathcal{I}$.

Lemma 2.2. *An ideal $\mathcal{I}$ has the approximation property if and only if there is a partition $\{C_\alpha \mid \alpha < \kappa\}$ of κ such that $\{D_\alpha \mid \alpha < \kappa\}$ is a basis for $\mathcal{I}$ and $D_\alpha = \bigcup_{\beta \leq \alpha} C_\beta$.*

Proof. If $\langle B_\alpha \mid \alpha < \kappa \rangle$ is an approximation sequence of $\mathcal{I}$, then it is a basis for $\mathcal{I}$. Moreover, let $C_0 = B_1$ and, for all $0 < \alpha < \kappa$, let $C_\alpha = B_{\alpha+1} \setminus \bigcup_{\beta<\alpha} C_\beta$. Clearly, $D_\alpha = B_{\alpha+1}$, thus $\{D_\alpha \mid \alpha < \kappa\}$ is a basis for $\mathcal{I}$.

Vice versa, let $\{C_\alpha \mid \alpha < \kappa\}$ be a partition of κ such that $\{D_\alpha \mid \alpha < \kappa\}$ is a basis for $\mathcal{I}$ and $D_\alpha = \bigcup_{\beta \leq \alpha} C_\beta$. Set $B_\alpha = \bigcup_{\beta<\alpha} D_\alpha$ for every $\alpha < \kappa$. Clearly $\langle B_\alpha \mid \alpha < \kappa \rangle$ is strictly increasing, and since $\{D_\alpha \mid \alpha < \kappa\}$ is a basis, for all $B \in \mathcal{I}$ there is $\alpha \in \kappa$ such that $B \subseteq D_\alpha \subseteq B_{\alpha+1}$, therefore $\mathcal{I}$ has the approximation property. □

It follows from Lemma 2.2 that $\mathcal{I}$ has the approximation property if and only if it admits a basis of size κ.

Let $\nu \in \{2, \kappa\}$ and consider the set ${}^\kappa\nu = \{x \mid x \colon \kappa \to \nu\}$. For any set $I \subseteq \kappa$, let $\mathrm{Fn}_I({}^\kappa\nu) = \{f \mid f \colon D \to \nu \text{ is a function, and } D \in I\}$. For $f \in \mathrm{Fn}_I({}^\kappa\nu)$, we define

$$\boldsymbol{N}_f({}^\kappa\nu) = \{x \in {}^\kappa\nu \mid f \subseteq x\}.$$

Definition 2.3. Let $\nu \in \{2, \kappa\}$. The **$\mathcal{I}$-topology** $\tau_\mathcal{I}$ on ${}^\kappa\nu$ is the topology generated by the collection

$$\mathcal{B}_\mathcal{I}({}^\kappa\nu) = \{\boldsymbol{N}_f({}^\kappa\nu) \mid f \in \mathrm{Fn}_\mathcal{I}({}^\kappa\nu)\}.$$

The family $\mathcal{B}_\mathcal{I}({}^\kappa\nu)$ is the **canonical basis** for $\tau_\mathcal{I}$ and its elements are called the (canonical) **basic $\mathcal{I}$-open sets** or **$\mathcal{I}$-cones**. The elements of $\tau_\mathcal{I}$ are also called **$\mathcal{I}$-open sets**, and similarly we will address closed (clopen) sets with respect to $\tau_\mathcal{I}$ as **$\mathcal{I}$-closed** (**$\mathcal{I}$-clopen**) sets. An **ideal topology** is an $\mathcal{I}$-topology for an ideal $\mathcal{I}$.

[1]A sequence $\langle s_\alpha \mid \alpha < \kappa \rangle \subseteq \kappa$ is strictly increasing if for every $\alpha < \beta < \kappa$, $s_\alpha \subsetneq s_\beta$, and it is continuous if for every limit ordinal $\alpha < \kappa$, $s_\alpha = \bigcup_{\beta<\alpha} s_\beta$.

When the space is clear from the context, we drop it from all the notation above.

Clearly, if $\mathcal{I} \subseteq \mathcal{I}'$ then $\tau_{\mathcal{I}} \subseteq \tau_{\mathcal{I}'}$. It is immediate that the NS_κ-topology refines the bounded topology. While the $\mathcal{I}$-topology coincides with the bounded topology when $\mathcal{I} = b$, it is strictly finer than τ_b whenever $\mathcal{I} \supsetneq b$. Indeed, for any unbounded set $D \in \mathcal{I}$ and any function $f : D \to 2$, the set $\boldsymbol{N}_f$ is $\mathcal{I}$-open and b-closed, but not b-open. As already mentioned in the introduction, when $\mathcal{I} = b$ we will refer to $({}^\kappa\kappa, \tau_b)$ and $({}^\kappa 2, \tau_b)$ as the generalized Baire space and the generalized Cantor space, respectively.

The following result is an easy observation about the size of the basic $\mathcal{I}$-open sets, which will be useful later on.

Lemma 2.4. *Let $\nu \in \{2, \kappa\}$. For every $f \in \mathrm{Fn}_{\mathcal{I}}({}^\kappa\nu)$, $|\boldsymbol{N}_f({}^\kappa\nu)| > \kappa$.*

Proof. Let $f \in \mathrm{Fn}_{\mathcal{I}}({}^\kappa\nu)$ with $D = \mathrm{dom}(f)$. Then, $|\boldsymbol{N}_f({}^\kappa\nu)| = 2^{|\kappa \setminus D|}$, so it is sufficient to show that $\kappa \setminus D$ is unbounded. Towards a contradiction, suppose that $|\kappa \setminus D| < \kappa$. Then, there exists $\gamma < \kappa$ such that $\kappa \setminus D \subseteq \gamma$. Since $\mathcal{I}$ extends the bounded ideal, $D \cup \gamma \in \mathcal{I}$. On the other hand, $D \cup \gamma = \kappa$ hence $\mathcal{I}$ is not proper, a contradiction. □

When $\mathcal{I} = \mathsf{NS}_\kappa$ and $\nu = 2$, Lemma 2.4 holds in a stronger form since any $\mathcal{I}$-cone with the induced topology is homeomorphic to the space $({}^\kappa 2, \tau_{\mathcal{I}})$ by [HKSW22, Lemma 1.6].

We recall that for a topological space (X, τ), the density character is the least possible cardinality of a dense subset of X, and the weight is the least possible cardinality of an open basis. Moreover, (X, τ) is κ-compact (or κ-Lindelöf) if every τ-open covering of X has a subcovering of size smaller than κ.

We say that a subset $A \subseteq {}^\kappa\nu$ is **$\mathcal{I}$-dense** if it is dense with respect to $\tau_{\mathcal{I}}$, i.e., if for every $\mathcal{I}$-open set $U \subseteq {}^\kappa\nu$, $A \cap U \neq \emptyset$.

The next lemma summarizes some properties of the ideal topology on ${}^\kappa 2$ and ${}^\kappa\kappa$. Some of these properties already appeared in [HKSW22], but we include them for the reader's convenience.

Lemma 2.5. *Consider the spaces $({}^\kappa 2, \tau_{\mathcal{I}})$ and $({}^\kappa\kappa, \tau_{\mathcal{I}})$. The following properties hold.*

1. *The topology $\tau_{\mathcal{I}}$ is perfect, regular Hausdorff, and zero-dimensional* [2].
2. *The topology $\tau_{\mathcal{I}}$ is closed under intersections of length at most α (for some ordinal α) if and only if $\alpha < \kappa$. Therefore, the collection of all $\mathcal{I}$-clopen subsets of ${}^\kappa\nu$ is a κ-algebra*[3].
3. *For all $x \in {}^\kappa\nu$, any $\mathcal{I}$-open neighborhood basis of x has size at least κ. Moreover, each point has an $\mathcal{I}$-open neighborhood basis of size κ if and only if $\mathcal{I}$ has a basis of size κ.*

Moreover, if $\mathcal{I}$ contains an unbounded subset of κ:

4. *The topology $\tau_{\mathcal{I}}$ is not compact nor κ-compact.*
5. *The topology $\tau_{\mathcal{I}}$ has weight 2^κ, and $|\tau_{\mathcal{I}}| = 2^{2^\kappa}$.*
6. *The topology $\tau_{\mathcal{I}}$ has density character 2^κ.*

[2]A topological space is perfect if it has no isolated points; it is zero-dimensional if it has a basis of clopen sets.

[3]Let γ be an ordinal. A γ-algebra on a set X is a family of subsets of X closed under the operations of complementation and well-ordered unions of length at most γ.

Proof. Let $\nu \in \{2, \kappa\}$.

(1) It is easy to check that the sets in $\mathcal{B}_\mathcal{I}$ are clopen, hence the topology $\tau_\mathcal{I}$ is zero-dimensional and regular. Moreover, it is perfect and Hausdorff, because $b \subseteq \mathcal{I}$ and τ_b is perfect and Hausdorff.

(2) Suppose $\alpha < \kappa$, let $\langle U_\beta \mid \beta < \alpha\rangle$ be a sequence of $\mathcal{I}$-open sets, and let $V = \bigcap_{\beta<\alpha} U_\beta$. To show that V is $\mathcal{I}$-open, for every $x \in V$ we construct $f \in \mathrm{Fn}_\mathcal{I}$ such that $x \in \boldsymbol{N}_f \subseteq V$. For every $\beta < \alpha$, let $D_\beta \in \mathcal{I}$ be such that $\boldsymbol{N}_{x\restriction D_\beta} \subseteq U_\beta$. Then $D = \bigcup_{\beta<\alpha} D_\beta \in \mathcal{I}$ by κ-completeness of $\mathcal{I}$, and $\boldsymbol{N}_{x\restriction D} \subseteq V$.

For all $A \subseteq \kappa$, let $\bar{0}_A$ be the function constant to 0 with domain A. Suppose $\alpha \geq \kappa$, and let $\langle D_\beta \mid \beta < \alpha\rangle$ be an increasing sequence of elements of $\mathcal{I}$ such that $\bigcup_{\beta<\alpha} D_\beta = \kappa \notin \mathcal{I}$. Set $V_\beta = \boldsymbol{N}_{\bar{0}_\kappa\restriction D_\beta}$. Then, $\bigcap_{\beta<\alpha} V_b = \{\bar{0}_\kappa\}$ is an $\mathcal{I}$-closed set and it is not $\mathcal{I}$-open.

(3) Let $x \in {}^\kappa\nu$ and $\mathcal{U}_\mathcal{I}$ be an arbitrary $\mathcal{I}$-open neighborhood basis of x. We want to construct a map $\mathcal{U}_\mathcal{I} \to \mathrm{Fn}_\mathcal{I}$ that sends each $U \in \mathcal{U}_\mathcal{I}$ to $f_U \subseteq x$ such that $\boldsymbol{N}_{f_U} \subseteq U$, yielding the neighborhood basis $\mathcal{U}' = \{\boldsymbol{N}_{f_U} \mid U \in \mathcal{U}_\mathcal{I}\}$. To do that, for every $U \in \mathcal{U}_\mathcal{I}$ choose $D_U \in \mathcal{I}$ such that $\boldsymbol{N}_{x\restriction D_U} \subseteq U$ and consider $\mathcal{U}' = \{\boldsymbol{N}_{x\restriction D_U} \mid U \in \mathcal{U}_\mathcal{I}\}$. Notice that the canonical map $\mathcal{U}_\mathcal{I} \to \mathcal{U}'$ is a surjection, hence for the first part it is enough to show that $|\mathcal{U}'| \geq \kappa$. Suppose, towards a contradiction, that $|\mathcal{U}'| < \kappa$. Since $\mathcal{I}$ is κ-complete, $\bar{D} = \bigcup_{U\in\mathcal{U}_\mathcal{I}} D_U \in \mathcal{I}$, so the $\mathcal{I}$-open neighborhood $\boldsymbol{N}_{x\restriction\bar{D}}$ does not contain any element of the open neighborhood basis $\mathcal{U}'$, a contradiction.

Next, we take care of the second part the statement. For the implication from left to right, we show that $\{D_U \mid U \in \mathcal{U}_\mathcal{I}\}$ is a basis for $\mathcal{I}$, so that $|\mathcal{U}_\mathcal{I}| = \kappa$ implies that $\mathcal{I}$ has a basis of size κ. Let $D \in \mathcal{I}$. Clearly, $\boldsymbol{N}_{x\restriction D}$ is an $\mathcal{I}$-open neighborhood of x, so there exists $U \in \mathcal{U}_\mathcal{I}$ such that $x \in U \subseteq \boldsymbol{N}_{x\restriction D}$. Thus $\boldsymbol{N}_{x\restriction D_U} \subseteq U \subseteq \boldsymbol{N}_{x\restriction D}$ and $D \subseteq D_U$. For the other direction, let $\mathcal{B}$ be a basis fo $\mathcal{I}$. It is enough to show that $\{\boldsymbol{N}_{x\restriction D} \mid D \in \mathcal{B}\}$ is a neighborhood basis of x. Let V be an $\mathcal{I}$-open set such that $x \in V$. There exists $f \in \mathrm{Fn}_\mathcal{I}$ such that $x \in \boldsymbol{N}_f \subseteq V$. Let $\mathrm{dom}(f) = D \in \mathcal{I}$, so there exists $D' \in \mathcal{B}$ such that $D \subseteq D'$. Therefore $x \in \boldsymbol{N}_{x\restriction D'} \subseteq \boldsymbol{N}_{x\restriction D} = \boldsymbol{N}_f$ and $\{\boldsymbol{N}_{x\restriction D} \mid D \in \mathcal{B}\}$ is a neighborhood basis of x.

Suppose now that $\mathcal{I}$ contains an unbounded subset D of κ. Consider the collection of functions $\mathcal{F}_D = \{f \mid f : D \to \nu\}$, and let $\mathcal{C}_D = \{\boldsymbol{N}_f \mid f \in \mathcal{F}_D\}$. Note that $\mathcal{C}_D$ is an $\mathcal{I}$-clopen partition of ${}^\kappa\nu$. Clearly, the sets in $\mathcal{C}_D$ are $\mathcal{I}$-clopen and pairwise disjoint. Moreover, for every $x \in {}^\kappa\nu$, $x \restriction D = f$ for some $f \in \mathcal{F}_D$, hence $x \in \boldsymbol{N}_f$.

(4) Since $\mathcal{C}_D$ is a $\mathcal{I}$-clopen partition of ${}^\kappa\nu$ of size 2^κ, no subcover of size $< \kappa$ can be extracted.

(5) Let $\mathcal{B}$ be an arbitrary basis for $\tau_\mathcal{I}$. Since $\mathcal{C}_D$ is a partition, the map

$$\mathcal{B} \to \mathcal{F} : B \mapsto \begin{cases} \emptyset & \text{if } \forall f \in \mathcal{F}_D (B \not\subseteq \boldsymbol{N}_f) \\ f & \text{if } B \subseteq \boldsymbol{N}_f \end{cases}$$

is a well-defined surjection. Since $|\mathcal{F}_D| = 2^\kappa$, $\mathcal{B} \geq 2^\kappa$. On the other hand, $\{\boldsymbol{N}_f | \, f \in \mathrm{Fn}_\mathcal{I}\}$ is a basis of size 2^κ, hence $\tau_\mathcal{I}$ has weight 2^κ.

Clearly, $|\tau_\mathcal{I}| \leq |\mathscr{P}(\mathcal{B})| = 2^{2^\kappa}$ as witnessed by the injective map $\tau_\mathcal{I} \to \mathscr{P}(\mathcal{B}) : U \mapsto \{B \in \mathcal{B} \mid B \subseteq U\}$. For the other inequality, for all $X \subseteq \mathcal{F}_D$ define $U_X = \bigcup_{f\in X} \boldsymbol{N}_f$. Notice that if $X, Y \subseteq \mathcal{F}_D$, then $X \neq Y$ implies $U_X \neq U_Y$. Thus, the map $\mathscr{P}(\mathcal{F}_D) \to \tau_\mathcal{I} : X \mapsto U_X$ is injective and we get $|\tau_\mathcal{I}| \geq 2^{2^\kappa}$.

(6) Let $E \subseteq {}^\kappa\nu$ be an $\mathcal{I}$-dense set. Since $\mathcal{C}_D$ is a partition of ${}^\kappa\nu$, the surjection $E \to \mathcal{F} : x \mapsto f$, where $f \in \mathcal{F}_D$ is (unique) such that $x \in \boldsymbol{N}_f$, witnesses $|E| \geq 2^\kappa$.

Moreover, for every $f \in \mathrm{Fn}_{\mathcal{I}}$ define the function f^0 as follows:

$$f^0(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma \in \mathrm{dom}(f) \\ 0 & \text{otherwise.} \end{cases}$$

The set $F^0 = \left\{ f^0 \mid f \in \mathrm{Fn}_{\mathcal{I}} \right\} \subseteq {}^{\kappa}\nu$ is $\mathcal{I}$-dense and has cardinality $|\,\mathrm{Fn}_{\mathcal{I}}\,| = 2^{\kappa}$. ☐

The next result, though not essential, is of independent interest.

Proposition 2.6. *Let $\nu \in \{2, \kappa\}$. The space $({}^{\kappa}\nu \times {}^{\kappa}\kappa, \tau_{\mathcal{I}} \times \tau_{\mathcal{I}})$ is homeomorphic to $({}^{\kappa}\kappa, \tau_{\mathcal{I}})$.*

Proof. Fix a bijection $\varphi : \nu \times \kappa \to \kappa$, and for every $\alpha < \kappa$, let $(\alpha)_1 < \nu$ and $(\alpha)_2 < \kappa$ such that $\varphi((\alpha)_1, (\alpha)_2) = \alpha$. Let $\Phi : {}^{\kappa}\nu \times {}^{\kappa}\kappa \to {}^{\kappa}\kappa$ be the function defined by $\Phi(x, y)(\alpha) = \varphi(x(\alpha), y(\alpha))$ for every $\alpha < \kappa$. We have that Φ is bijective because φ is a bijection. We claim that Φ is an homeomorphism. To see that Φ is continuous, note that for any $f \in \mathrm{Fn}_{\mathcal{I}}({}^{\kappa}\kappa)$, if $D = \mathrm{dom}(f)$,

$$\Phi^{-1}(\boldsymbol{N}_f) = \{(x, y) \mid \forall \alpha \in D\, (x(\alpha) = (f(\alpha))_1 \wedge y(\alpha) = (f(\alpha))_2)\} = \boldsymbol{N}_{f_1} \times \boldsymbol{N}_{f_2}.$$

where $f_1, f_2 \in \mathrm{Fn}_{\mathcal{I}}$ are defined by $f_1(\alpha) = (f(\alpha))_1$ and $f_2(\alpha) = (f(\alpha))_2$ for every $\alpha \in D$. To see that Φ is open, for any $f, g \in \mathrm{Fn}_{\mathcal{I}}$, if $D_f = \mathrm{dom}(f)$ and $D_g = \mathrm{dom}(g)$, notice that $\Phi(\boldsymbol{N}_f \times \boldsymbol{N}_g) = \boldsymbol{N}_h$, where $h : D_f \cap D_g \to \kappa$ is defined by $h(\alpha) = \varphi(f(\alpha), g(\alpha))$ for every $\alpha \in D_f \cap D_g$. Clearly, $D_f \cap D_g \in \mathcal{I}$. ☐

3. The $\mathcal{I}$-continuous functions

Given $\nu, \mu \in \{2, \kappa\}$, a function $\Phi\colon {}^{\kappa}\nu \to {}^{\kappa}\mu$ is **$\mathcal{I}$-continuous** if it continuous with respect to $\tau_{\mathcal{I}}$, i.e., $f\colon ({}^{\kappa}\nu, \tau_{\mathcal{I}}) \to ({}^{\kappa}\mu, \tau_{\mathcal{I}})$ is continuous.

Throughout this section, let $\nu \in \{2, \kappa\}$. Similarly to classical setting (see [Kec95, Proposition 2.6]) and to the generalized setting with respect to the bounded topology τ_b (see [AMR22, Definition 3.6]), every $\mathcal{I}$-continuous function can be represented by a monotone and domain-increasing function $\varphi\colon \mathrm{Fn}_{\mathcal{I}}({}^{\kappa}\nu) \to \mathrm{Fn}_{\mathcal{I}}({}^{\kappa}\nu)$. However, when dealing with ideals $\mathcal{I}$ strictly extending the bounded ideal, the definition is rather different, as limits over direct sets are necessary.

Let X be a topological space, and let $(\mathcal{D}, \leq)$ be a directed set, still denoted by $\mathcal{D}$. A point $x \in X$ is a $\mathcal{D}$-limit of a $\mathcal{D}$-net $(x_d)_{d \in \mathcal{D}}$ of points from X if for every open neighborhood U of x there is $d \in \mathcal{D}$ such that $x_{d'} \in U$ for all $d' \in \mathcal{D}$ with $d' \geq d$. When this happens, we write $x = \lim_{d \in \mathcal{D}} x_d$. We will consider $\mathcal{D}$-limits where $\mathcal{D} = (\mathcal{I}, \subseteq)$.

Definition 3.1. Let $\varphi\colon \mathrm{Fn}_{\mathcal{I}}({}^{\kappa}\nu) \to \mathrm{Fn}_{\mathcal{I}}({}^{\kappa}\nu)$ be a function. We say that φ is **monotone** if $f \subseteq g$ implies $\varphi(f) \subseteq \varphi(g)$ for all $f, g \in \mathrm{Fn}_{\mathcal{I}}({}^{\kappa}\nu)$.

We say that φ is **continuous** if it is monotone and for all $x \in {}^{\kappa}\nu$ and $D \in \mathcal{I}$ there exists $E \in \mathcal{I}$ such that $D \subseteq \mathrm{dom}(\varphi(x \restriction E))$.

Let $\varphi\colon \mathrm{Fn}_{\mathcal{I}}({}^{\kappa}\nu) \to \mathrm{Fn}_{\mathcal{I}}({}^{\kappa}\nu)$ be a continuous function. The function induced by φ is $\varphi^*\colon {}^{\kappa}\nu \to {}^{\kappa}\nu$ defined for every $x \in {}^{\kappa}\nu$ by

$$\varphi^*(x) = \lim_{D \in \mathcal{I}} \varphi(x \restriction D). \tag{3.1}$$

Note that since φ is continuous, the limit in (3.1) exists because for all $D \in \mathcal{I}$ there exists $E \in \mathcal{I}$ such that

$$\varphi(x \restriction E') \restriction D = \varphi(x \restriction E) \restriction D$$

for all $E \subseteq E'$ with $E' \in \mathcal{I}$. Moreover, such limit is unique since ${}^{\kappa}\nu$ is Hausdorff by Lemma 2.5(1), and therefore φ^* is well-defined.

Notice that if φ is continuous, then $\lim_{D \in \mathcal{I}} \varphi(x \upharpoonright D) = \lim_{D \in \mathcal{B}} \varphi(x \upharpoonright D)$ for any basis $\mathcal{B}$ for $\mathcal{I}$. In particular, when $\mathcal{I}$ is the bounded ideal, $\varphi^*(x) = \lim_{\alpha < \kappa} \varphi(x \upharpoonright \alpha)$ is the usual definition also adopted in [AMR22, Definition 3.6].

Proposition 3.2. *Let* $\varphi\colon \mathrm{Fn}_{\mathcal{I}}({}^{\kappa}\nu) \to \mathrm{Fn}_{\mathcal{I}}({}^{\kappa}\nu)$ *be a continuous function. Then,* $\varphi^*\colon {}^{\kappa}\nu \to {}^{\kappa}\nu$ *is* $\mathcal{I}$*-continuous.*

Proof. It is sufficient to check that for every $f \in \mathrm{Fn}_{\mathcal{I}}$,

$$(\varphi^*)^{-1}(\boldsymbol{N}_f) = \bigcup \{\boldsymbol{N}_g \mid f \subseteq \varphi(g)\}.$$

Since $\bigcup \{\boldsymbol{N}_g \mid f \subseteq \varphi(g)\}$ is $\mathcal{I}$-open for every $f \in \mathrm{Fn}_{\mathcal{I}}$, φ^* is $\mathcal{I}$-continuous.

First, assume that $x \in \bigcup \{\boldsymbol{N}_g \mid f \subseteq \varphi(g)\}$ and let $D = \mathrm{dom}(f)$. Our goal is to show that $\varphi^*(x) \upharpoonright D = f$. Let $g \in \mathrm{Fn}_{\mathcal{I}}$ such that $g \subseteq x$ and $f \subseteq \varphi(g)$. If $E = \mathrm{dom}(g)$, $g = x \upharpoonright E$ and $f = \varphi(x \upharpoonright E) \upharpoonright D$. Since φ is monotone, for every $E' \in \mathcal{I}$ such that $E \subseteq E'$ we have $\varphi(x \upharpoonright E) \upharpoonright D = \varphi(x \upharpoonright E') \upharpoonright D$. By definition of φ^*, given $\boldsymbol{N}_{\varphi^*(x) \upharpoonright D}$ there exists $F \in \mathcal{I}$ for every $F' \in \mathcal{I}$ with $F \subseteq F'$, $\varphi^*(x) \upharpoonright D = \varphi(x \upharpoonright F') \upharpoonright D$. Let $G = E \cup F \in \mathcal{I}$. Then, for every $G' \in \mathcal{I}$ such that $G \subseteq G'$ we have

$$\varphi^*(x) \upharpoonright D = \varphi(x \upharpoonright G') \upharpoonright D = f.$$

Hence $x \in (\varphi^*)^{-1}(\boldsymbol{N}_f)$.

Next, assume that $x \in (\varphi^*)^{-1}(\boldsymbol{N}_f)$. Then, $\varphi^*(x) \in \boldsymbol{N}_f$ and, if $D = \mathrm{dom}(f)$, $\varphi^*(x) \upharpoonright D = f$. Since φ is continous, there exists $E \in \mathcal{I}$ such that $D \subseteq \mathrm{dom}(\varphi(x \upharpoonright E))$. Therefore, by definition of φ^*, $\varphi(x \upharpoonright E) \upharpoonright D = \varphi^*(x) \upharpoonright D$. Let $g = x \upharpoonright E$ to get $x \in \bigcup \{\boldsymbol{N}_g \mid f \subseteq \varphi(g)\}$. □

Proposition 3.3. *Let* $\Phi\colon {}^{\kappa}\nu \to {}^{\kappa}\nu$ *be an* $\mathcal{I}$*-continuous function. Then, there is a continuous* $\varphi\colon \mathrm{Fn}_{\mathcal{I}}({}^{\kappa}\nu) \to \mathrm{Fn}_{\mathcal{I}}({}^{\kappa}\nu)$ *such that* $\varphi^* = \Phi$.

Proof. For every $g \in \mathrm{Fn}_{\mathcal{I}}$, we define $X_g = \{f \in \mathrm{Fn}_{\mathcal{I}} \mid \Phi(\boldsymbol{N}_g) \subseteq \boldsymbol{N}_f\}$. Note that X_g is a lattice and every chain $\mathcal{C}$ in X_g has an upper bound (it is sufficient to take $G = \bigcup_{f \in \mathcal{C}} f$, since $\Phi(\boldsymbol{N}_g) \subseteq \bigcap_{f \in \mathcal{C}} \boldsymbol{N}_f = \boldsymbol{N}_G$). By Zorn's lemma, X_g has at least one maximal element. We now show that this maximal element is unique. Assume that $f, f' \in X_g$ are both maximal. Since $\Phi\big(\boldsymbol{N}_g\big) \subseteq \boldsymbol{N}_f \cap \boldsymbol{N}_{f'}$, f and f' must be compatible. Therefore, in order not to violate the maximality of either one, we must have $f = f'$. For every $g \in \mathrm{Fn}_{\mathcal{I}}$, let $\varphi(g)$ be the (unique) maximal $f \in X_g$.

Let us show that the function φ is monotone. Assume that $g \subseteq g' \in \mathrm{Fn}_{\mathcal{I}}$ and let $E = \mathrm{dom}(g)$ and $E' = \mathrm{dom}(g')$. Since $g \subseteq g'$, $\boldsymbol{N}_{g'} \subseteq \boldsymbol{N}_g$ and $\Phi(\boldsymbol{N}_{g'}) \subseteq \Phi(\boldsymbol{N}_g)$. From the definition of Φ we know that $\Phi(\boldsymbol{N}_g) \subseteq \boldsymbol{N}_{\varphi(g)}$. Therefore $\Phi(\boldsymbol{N}_{g'}) \subseteq \boldsymbol{N}_{\varphi(g)}$ and $\varphi(g) \in X_{g'}$. Thus $\varphi(g) \subseteq \varphi(g')$, by maximality of $\varphi(g')$.

It remains to prove that φ is continuous and that for every $x \in {}^{\kappa}\nu$, $\varphi^*(x) = \Phi(x)$. Fix $x \in {}^{\kappa}\nu$. By the $\mathcal{I}$-continuity of Φ, for every $D \in \mathcal{I}$ there exists $D' \in \mathcal{I}$ such that $\Phi(\boldsymbol{N}_{x \upharpoonright D'}) \subseteq \boldsymbol{N}_{\Phi(x) \upharpoonright D}$. Let $E = D \cup D' \in \mathcal{I}$, so $\boldsymbol{N}_{x \upharpoonright E} \subseteq \boldsymbol{N}_{x \upharpoonright D'}$ and $\Phi(\boldsymbol{N}_{x \upharpoonright E}) \subseteq \Phi(\boldsymbol{N}_{x \upharpoonright D'})$. Then, $\Phi(\boldsymbol{N}_{x \upharpoonright E}) \subseteq \boldsymbol{N}_{\Phi(x) \upharpoonright D}$ and $\Phi(x) \upharpoonright D \in X_{x \upharpoonright E}$. By maximality of $\varphi(x \upharpoonright E)$, $\Phi(x) \upharpoonright D \subseteq \varphi(x \upharpoonright E)$. Thus, we have proved that for every $D \in \mathcal{I}$ there exists $E \in \mathcal{I}$ such that $\Phi(x) \upharpoonright D = \varphi(x \upharpoonright E) \upharpoonright D$. Finally, $\varphi^*(x) = \Phi(x)$ due to the uniqueness of the limit. □

One could be tempted to only require that φ is monotone and that $\bigcup_{D\in\mathcal{I}}\varphi(x\restriction D)\in {}^\kappa\nu$ for every $x\in {}^\kappa\nu$. However, the function $\varphi^*(x)=\bigcup_{D\in\mathcal{I}}\varphi(x\restriction D)$ may fail to be continuous, as the next example shows.

For all $A\subseteq\kappa$, let $\bar{\beta}_A$ be the function constant to β with domain A.

Example 3.4. Let $D\subseteq\kappa$ be the subset of odd[4] ordinals, and let $\mathcal{I}$ the ideal generated by the bounded ideal together with D. Note that $E\in\mathcal{I}$ if and only if $E\setminus D$ is bounded.

Define $\varphi\colon \mathrm{Fn}_{\mathcal{I}}({}^\kappa\nu)\to\mathrm{Fn}_{\mathcal{I}}({}^\kappa\nu)$ as follows. For any $f\in\mathrm{Fn}_{\mathcal{I}}({}^\kappa\nu)$, and for every $\alpha\in\mathrm{dom}(f)\setminus D$, set:

- $\varphi(f)(\alpha)=f(\alpha)$,
- $\varphi(f)(\alpha+1)=f(\alpha)$.

Note that $\mathrm{dom}(\varphi(f))=(\mathrm{dom}(f)\setminus D)\cup\{\alpha+1\mid\alpha\in\mathrm{dom}(f)\setminus D\}$, and since $\mathrm{dom}(f)\in\mathcal{I}$, we get that $\mathrm{dom}(f)\setminus D$ is bounded. Therefore, for every $f\in\mathrm{Fn}_{\mathcal{I}}({}^\kappa\nu)$, $\mathrm{dom}(\varphi(f))$ is bounded.

It is easy to verify that φ is monotone and that for every $x\in{}^\kappa\nu$, $\varphi^*(x)=\bigcup_{D\in\mathcal{I}}\varphi(x\restriction D)\in{}^\kappa\nu$. To see this, suppose, towards a contradiction, that there are $E,E'\in\mathcal{I}$ and $\alpha\in\mathrm{dom}(\varphi(x\restriction E))\cap\mathrm{dom}(\varphi(x\restriction E'))$ such that $\varphi(x\restriction E)(\alpha)\neq\varphi(x\restriction E')(\alpha)$. Then either $\alpha\notin D$ or $\alpha\in D$. If $\alpha\notin D$, then by the definition of φ, $\varphi(x\restriction E)(\alpha)=x(\alpha)=\varphi(x\restriction E')(\alpha)$, a contradiction. If $\alpha\in D$, then there is $\beta\notin D$ such that $\alpha=\beta+1$ and $\beta\in\mathrm{dom}(\varphi(x\restriction E))\cap\mathrm{dom}(\varphi(x\restriction E'))$. By the definition of φ, $\varphi(x\restriction E)(\alpha)=x(\beta)=\varphi(x\restriction E')(\alpha)$, a contradiction.

However, φ^* is not $\mathcal{I}$-continuous. To see this, let $f=\bar{0}_D$. We will show that $(\varphi^*)^{-1}(\boldsymbol{N}_f)$ is not open.

Notice that $\varphi^*(\bar{0}_\kappa)=\bigcup_{E\in\mathcal{B}}\varphi(\bar{0}_\kappa\restriction E)=\bar{0}_\kappa$. Thus, $\bar{0}_\kappa\in(\varphi^*)^{-1}(\boldsymbol{N}_f)$. It remains to verify that for all $E\in\mathcal{I}$, $\boldsymbol{N}_{\bar{0}_\kappa\restriction E}\not\subseteq(\varphi^*)^{-1}(\boldsymbol{N}_f)$.

Fix $E\in\mathcal{I}$ and let $x=\bar{0}_E\cup\bar{1}_{\kappa\setminus E}$. Since $\bar{0}_E=\bar{0}_\kappa\restriction E$, $x\in\boldsymbol{N}_{\bar{0}_\kappa\restriction E}$. We have that $\kappa\setminus(D\cup E)\neq\emptyset$ (otherwise $\kappa=E\cup D\in\mathcal{I}$ against $\mathcal{I}$ being proper) so let $\alpha\in\kappa\setminus(D\cup E)$. Since $\alpha\in\mathrm{dom}(x\restriction E\cup\alpha+1)\setminus D$, $\alpha+1\in\mathrm{dom}(\varphi(x\restriction E\cup\alpha+1))$ and

$$\varphi(x\restriction E\cup\alpha+1)(\alpha+1)=(x\restriction E\cup\alpha+1)(\alpha)=x(\alpha)=1.$$

Then, $\varphi^*(x)(\alpha+1)=\varphi(x\restriction E\cup\alpha+1)(\alpha+1)=1$. Finally, notice that $\alpha+1\in D$, thus $f(\alpha+1)=0\neq1=\varphi^*(x)(\alpha+1)$. So $x\notin(\varphi^*)^{-1}(\boldsymbol{N}_f)$.

4. The $\mathcal{I}$-Borel sets and their hierarchy

A **pointclass $\boldsymbol{\Gamma}$** is a class-function assigning to every nonempty topological space X a nonempty family $\boldsymbol{\Gamma}(X)\subseteq\mathscr{P}(X)$. The dual $\check{\boldsymbol{\Gamma}}$ of $\boldsymbol{\Gamma}$ is defined by $\check{\boldsymbol{\Gamma}}(X)=\{X\setminus A\mid A\in\boldsymbol{\Gamma}(X)\}$. A pointclass $\boldsymbol{\Gamma}$ is **boldface** if it is closed under continuous preimages, that is, if $f\colon X\to Y$ is continuous and $B\in\boldsymbol{\Gamma}(Y)$ implies $f^{-1}(B)\in\boldsymbol{\Gamma}(X)$. A pointclass $\boldsymbol{\Gamma}$ is **hereditary** if $\boldsymbol{\Gamma}(Y)=\{A\cap Y\mid A\in\boldsymbol{\Gamma}(X)\}$ for every $Y\subseteq X$. Notice that if $\boldsymbol{\Gamma}$ is a boldface pointclass, then it is hereditary if and only if for every $Y\subseteq X$ and $B\in\boldsymbol{\Gamma}(Y)$ there is $A\in\boldsymbol{\Gamma}(X)$ such that $B=A\cap Y$. A set $\mathcal{U}\subseteq Y\times X$ is **Y-universal for $\boldsymbol{\Gamma}(X)$** if $\mathcal{U}\in\boldsymbol{\Gamma}(Y\times X)$ and $\boldsymbol{\Gamma}(X)=\{\mathcal{U}_y\mid y\in Y\}$, where $\mathcal{U}_y=\{x\in X\mid(y,x)\in\mathcal{U}\}$. It is clear that if $\mathcal{U}$ is Y-universal for $\boldsymbol{\Gamma}(X)$, then its complement is Y-universal for $\check{\boldsymbol{\Gamma}}(X)$.

[4]Recall that every ordinal α can be written uniquely as $\alpha=\gamma+n$, with $n<\omega$ and either $\gamma=0$ or γ limit. Accordingly, we say that γ is even (respectively, odd) if n is even (respectively, odd).

We recall that a κ^+-algebra on set X is a family of subsets of X closed under the operations of complementation and unions of length at most κ. When X is a topological space, the κ^+-algebra generated by the topology of X, denoted by κ^+-$\mathbf{Bor}(X)$, the smallest κ^+-algebra on X containing all its open sets.

Let $\nu \in \{2, \kappa\}$, and consider the topological space $({}^\kappa\nu, \tau_{\mathcal{I}})$. We denote the κ^+-algebra generated by $\tau_{\mathcal{I}}$ by $\mathcal{I}$-$\mathbf{Bor}({}^\kappa\nu)$, i.e., $\mathcal{I}$-$\mathbf{Bor}({}^\kappa\nu) = \kappa^+$-$\mathbf{Bor}({}^\kappa\nu, \tau_{\mathcal{I}})$, and we call its elements **$\mathcal{I}$-Borel** sets. When $\mathcal{I} = b$, $\mathcal{I}$-$\mathbf{Bor}({}^\kappa\nu) = \kappa^+$-$\mathbf{Bor}({}^\kappa\nu, \tau_b)$ are the usual generalized Borel sets extensively studied in [MV93, FHK14, ACRP25].

As already observed in [ACRP25], for every topological space X, the κ^+-algebra generated by any topology on X can be stratified in a hierarchy (called the κ^+-Borel hierarchy) formed by the classes κ^+-$\mathbf{\Sigma}^0_\alpha(X)$, κ^+-$\mathbf{\Pi}^0_\alpha(X)$, and κ^+-$\mathbf{\Delta}^0_\alpha(X)$, where α ranges over nonzero ordinals. The pointclasses κ^+-$\mathbf{Bor}$, κ^+-$\mathbf{\Sigma}^0_\alpha$, κ^+-$\mathbf{\Pi}^0_\alpha$ and κ^+-$\mathbf{\Delta}^0_\alpha$ are boldface, and κ^+-$\mathbf{Bor}$, κ^+-$\mathbf{\Sigma}^0_\alpha$, κ^+-$\mathbf{\Pi}^0_\alpha$ are also hereditary. In our case study $X = ({}^\kappa\nu, \tau_{\mathcal{I}})$, the hierarchy of $\mathcal{I}$-Borel sets is defined as follows, and we call it the **$\mathcal{I}$-Borel hierarchy**.

Definition 4.1. The following classes are defined by recursion on the ordinal $\alpha \geq 1$:

$$\mathcal{I}\text{-}\mathbf{\Sigma}^0_1({}^\kappa\nu) = \{U \subseteq {}^\kappa\nu \mid U \text{ is } \mathcal{I}\text{-open }\} \qquad \mathcal{I}\text{-}\mathbf{\Pi}^0_1({}^\kappa\nu) = \{C \subseteq {}^\kappa\nu \mid C \text{ is } \mathcal{I}\text{-closed }\}$$

$$\mathcal{I}\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\nu) = \left\{ \bigcup_{\gamma<\kappa} A_\gamma \mid A_\gamma \in \bigcup_{1\leq\beta<\alpha} \mathcal{I}\text{-}\mathbf{\Pi}^0_\beta({}^\kappa\nu) \right\} \quad \mathcal{I}\text{-}\mathbf{\Pi}^0_\alpha({}^\kappa\nu) = \left\{ X \setminus A \mid A \in \mathcal{I}\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\nu) \right\}.$$

We also set $\mathcal{I}$-$\mathbf{\Delta}^0_\alpha({}^\kappa\nu) = \mathcal{I}$-$\mathbf{\Sigma}^0_\alpha({}^\kappa\nu) \cap \mathcal{I}$-$\mathbf{\Pi}^0_\alpha({}^\kappa\nu)$.

Note that, to be coherent with [ACRP25] and with the usual notation of generalized Borel hierarchy, one should write the classes in Definition 4.1 as κ^+-$\mathbf{\Sigma}^0_\alpha({}^\kappa\nu, \tau_{\mathcal{I}})$, κ^+-$\mathbf{\Pi}^0_\alpha({}^\kappa\nu, \tau_{\mathcal{I}})$, and κ^+-$\mathbf{\Delta}^0_\alpha({}^\kappa\nu, \tau_{\mathcal{I}})$. However, for simplicity of notation, we decide to adopt the present one. When the space is clear from the context, we drop it from all the notation above.

It is easy to check that

$$\mathcal{I}\text{-}\mathbf{Bor}({}^\kappa\nu) = \bigcup_{1\leq\alpha<\kappa^+} \mathcal{I}\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\nu) = \bigcup_{1\leq\alpha<\kappa^+} \mathcal{I}\text{-}\mathbf{\Pi}^0_\alpha({}^\kappa\nu) = \bigcup_{1\leq\alpha<\kappa^+} \mathcal{I}\text{-}\mathbf{\Delta}^0_\alpha({}^\kappa\nu). \tag{4.1}$$

Since the ideal topology $\tau_{\mathcal{I}}$ is finer than the bounded topology τ_b, a straightforward induction shows the following relations between the classes in the $\mathcal{I}$-Borel and the κ^+-Borel hierarchy.

Proposition 4.2. *For every $\alpha < \kappa^+$, κ^+-$\mathbf{\Sigma}^0_\alpha({}^\kappa\nu, \tau_b) \subseteq \mathcal{I}$-$\mathbf{\Sigma}^0_\alpha({}^\kappa\nu)$, hence κ^+-$\mathbf{\Pi}^0_\alpha({}^\kappa\nu, \tau_b) \subseteq \mathcal{I}$-$\mathbf{\Pi}^0_\alpha({}^\kappa\nu)$ and κ^+-$\mathbf{\Delta}^0_\alpha({}^\kappa\nu, \tau_b) \subseteq \mathcal{I}$-$\mathbf{\Delta}^0_\alpha({}^\kappa\nu)$. Therefore, κ^+-$\mathbf{Bor}({}^\kappa\nu, \tau_b) \subseteq \mathcal{I}$-$\mathbf{Bor}({}^\kappa\nu)$.*

Proof. We show that κ^+-$\mathbf{\Sigma}^0_\alpha({}^\kappa\nu, \tau_b) \subseteq \mathcal{I}$-$\mathbf{\Sigma}^0_\alpha({}^\kappa\nu)$ by induction on $\alpha < \kappa^+$. Since $b \subseteq \mathcal{I}$, $\tau_b \subseteq \tau_{\mathcal{I}}$. Assume $\alpha > 1$ and that κ^+-$\mathbf{\Sigma}^0_\beta({}^\kappa\nu, \tau_b) \subseteq \mathcal{I}$-$\mathbf{\Sigma}^0_\beta({}^\kappa\nu)$ for every $\beta < \alpha$. If $A \in \kappa^+$-$\mathbf{\Sigma}^0_\alpha({}^\kappa\nu, \tau_b)$, then $A = \bigcup_{i<\kappa} A_i$ with $A_i \in \bigcup_{\beta<\alpha} \kappa^+$-$\mathbf{\Pi}^0_\beta({}^\kappa\nu, \tau_b)$. Since κ^+-$\mathbf{\Pi}^0_\beta({}^\kappa\nu, \tau_b) \subseteq \mathcal{I}$-$\mathbf{\Pi}^0_\beta({}^\kappa\nu)$, by the induction hypothesis, $A_i \in \bigcup_{\beta<\alpha} \mathcal{I}$-$\mathbf{\Pi}^0_\beta({}^\kappa\nu)$ for every $i < \kappa$. Hence $A \in \mathcal{I}$-$\mathbf{\Sigma}^0_\alpha({}^\kappa\nu)$. □

Recall that $\mathcal{B}_{\mathcal{I}}$ denotes the canonical basis for $\tau_{\mathcal{I}}$. The following remark shows that $\mathcal{B}_{\mathcal{I}}({}^\kappa\nu) \subseteq \kappa^+$-$\mathbf{\Pi}^0_1({}^\kappa\nu, \tau_b)$.

Remark 4.3. Let $f \in \mathrm{Fn}_{\mathcal{I}}$ and let $D = \mathrm{dom}(f)$. Then, $\boldsymbol{N}_f = \bigcap_{\alpha \in D} \mathcal{N}^f_\alpha$, where $\mathcal{N}^f_\alpha = \{x \in {}^\kappa\nu \mid x(\alpha) = f(\alpha)\}$. Since $\mathcal{N}^f_\alpha \in \kappa^+\text{-}\boldsymbol{\Delta}^0_1({}^\kappa\nu, \tau_b)$ and $|D| \leq \kappa$, $\boldsymbol{N}_f \in \kappa^+\text{-}\boldsymbol{\Pi}^0_1({}^\kappa\nu, \tau_b)$. Thus for every $f \in \mathrm{Fn}_{\mathcal{I}}$, $\boldsymbol{N}_f \in \kappa^+\text{-}\boldsymbol{\Pi}^0_1({}^\kappa\nu, \tau_b)$.

We say that the $\mathcal{I}$-Borel hierarchy is **increasing** if for every $1 \leq \alpha < \beta$ we have $\mathcal{I}\text{-}\boldsymbol{\Sigma}^0_\alpha({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\boldsymbol{\Sigma}^0_\beta({}^\kappa\nu)$. In this respect, the only problematic case is when $\alpha = 1$ and $\beta = 2$, as already noticed more in general in [AMR22, Lemma 2.2] for any topological space (X, τ). Indeed, $\mathcal{I}\text{-}\boldsymbol{\Sigma}^0_\alpha({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\boldsymbol{\Sigma}^0_\beta({}^\kappa\nu)$ for every $2 \leq \alpha < \beta$, and $\mathcal{I}\text{-}\boldsymbol{\Sigma}^0_1({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\boldsymbol{\Pi}^0_2({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\boldsymbol{\Sigma}^0_3({}^\kappa\nu)$. In particular, the $\mathcal{I}$-Borel hierarchy is increasing if and only if $\mathcal{I}\text{-}\boldsymbol{\Sigma}^0_1({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\boldsymbol{\Sigma}^0_2({}^\kappa\nu)$.

It was already observed in [HKSW22] that $\mathcal{I}\text{-}\boldsymbol{\Sigma}^0_1({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\boldsymbol{\Sigma}^0_2({}^\kappa\nu)$ when $\mathcal{I}$ has a basis of size κ.

Proposition 4.4. [HKSW22, Proposition 1.5] *Assume that $\mathcal{I}$ has a basis of size κ. Then every $\mathcal{I}$-open subset of ${}^\kappa\nu$ is the union of κ-many $\mathcal{I}$-clopen sets, therefore $\mathcal{I}\text{-}\boldsymbol{\Sigma}^0_1({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\boldsymbol{\Sigma}^0_2({}^\kappa\nu)$.*

Proof. Let $\mathcal{B} = \{D_i \mid i < \kappa\}$ be a basis for $\mathcal{I}$ and let $U \subseteq {}^\kappa\nu$ be $\mathcal{I}$-open. For every $x \in U$, there exists $B_x \in \mathcal{B}$ such that $\boldsymbol{N}_{x \restriction B_x} \subseteq U$. Clearly $U = \bigcup_{x \in U} \boldsymbol{N}_{x \restriction B_x}$. Now, for every $i < \kappa$, let

$$O_i = \bigcup \{\boldsymbol{N}_{x \restriction B_x} \mid x \in U \wedge B_x = D_i\}.$$

If $y \notin O_i$, then $\boldsymbol{N}_{y \restriction D_i} \cap O_i = \emptyset$. Thus O_i is $\mathcal{I}$-clopen and $U = \bigcup_{i<\kappa} O_i$. □

Corollary 4.5. *Assume that $\mathcal{I}$ has a basis of size κ. Then, the $\mathcal{I}$-Borel hierarchy on ${}^\kappa\nu$ is increasing.*

As soon as we remove the assumption on the size of the basis of $\mathcal{I}$, the $\mathcal{I}$-Borel hierarchy may not be increasing. For example, if $\mathcal{I} = \mathsf{NS}_\kappa$ and we consider the space ${}^\kappa 2$, the set $ub_\kappa = \{x \in {}^\kappa 2 \mid x = \chi_A$[5] for some $A \subseteq \kappa$ unbounded$\}$ is $\mathcal{I}$-open but not $\mathcal{I}\text{-}\boldsymbol{\Sigma}^0_2({}^\kappa 2)$ (see [HKSW22, Corollary 3.9 and Theorem 3.10]).

Note however that certain inclusions hold regardless of whether the $\mathcal{I}$-Borel hierarchy is increasing or not. In particular, for all ordinals $\alpha \leq \beta$ we have $\mathcal{I}\text{-}\boldsymbol{\Sigma}^0_\alpha({}^\kappa\nu) \cup \mathcal{I}\text{-}\boldsymbol{\Pi}^0_\alpha({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\boldsymbol{\Sigma}^0_\beta({}^\kappa\nu) \cup \mathcal{I}\text{-}\boldsymbol{\Pi}^0_\beta({}^\kappa\nu)$ and $\mathcal{I}\text{-}\boldsymbol{\Delta}^0_\alpha({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\boldsymbol{\Delta}^0_\beta({}^\kappa\nu)$.

Remark 4.6. The set $ub_\kappa \subseteq {}^\kappa 2$ is $\mathcal{I}$-open if and only if $\mathcal{I}$ is tall by [HKSW22, Corollary 3.9]. Since $ub_\kappa \notin \mathcal{I}\text{-}\boldsymbol{\Sigma}^0_2({}^\kappa 2)$ for every ideal $\mathcal{I}$ by [HKSW22, Theorem 3.10], Proposition 4.4 implies that no ideal $\mathcal{I}$ can simultaneously be tall and have a basis of cardinality κ. Consequently, an ideal has the approximation property if and only if it is not tall.

We now turn to the question of whether the $\mathcal{I}$-Borel hierarchy collapses, as a fundamental parameter measuring the behavior of the $\mathcal{I}$-Borel hierarchy is its length. From (4.1), we know that an upper bound for its length is κ^+, but it remained unknown until now whether the $\mathcal{I}$-Borel hierarchy on ${}^\kappa\nu$ could be strictly shorter. We say that the $\mathcal{I}$-Borel hierarchy on ${}^\kappa\nu$ **collapses** if

$$\min\left\{\alpha \in \mathrm{Ord} \mid \mathcal{I}\text{-}\boldsymbol{\Sigma}^0_\alpha({}^\kappa\nu) = \mathcal{I}\text{-}\mathbf{Bor}({}^\kappa\nu)\right\} < \kappa^+.$$

[5]For any $A \subseteq \kappa$, $\chi_A \colon \kappa \to 2$ is the function satisfying $\chi_A(\alpha) = 1 \Leftrightarrow \alpha \in A$.

In the next results we show that for *any* ideal $\mathcal{I}$, the $\mathcal{I}$-Borel hierarchy has length κ^+, that is, $\mathcal{I}\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\nu) \subsetneq \mathcal{I}\text{-}\mathbf{\Sigma}^0_\beta({}^\kappa\nu)$ for all $1 \leq \alpha < \beta < \kappa^+$, and therefore it does not collapse. This answers [HKSW22, Question 1].

In the generalized context with $\mathcal{I} = b$, the non-collapse of the κ^+-Borel hierarchy on $({}^\kappa 2, \tau_b)$ was first established in [AMR22, Proposition 4.19] for arbitrary infinite cardinals κ, without assuming $2^{<\kappa} = \kappa$. As shown in Theorem 4.7, this implies that the κ^+-Borel hierarchy does not collapse on any space containing a homeomorphic copy of $({}^\kappa 2, \tau_b)$. This result is well-known, but we include the proof for the reader's convenience.

Theorem 4.7. *For any topological space X containing an homeomorphic copy of $({}^\kappa 2, \tau_b)$, the κ^+-Borel hierarchy on X does not collapse.*

Proof. Let $f : {}^\kappa 2 \to X$ be a topological embedding and $Y = \operatorname{ran}(f) \subseteq X$. Suppose, towards a contradiction, that $\kappa^+\text{-}\mathbf{\Sigma}^0_\alpha(X) = \kappa^+\text{-}\mathbf{Bor}(X)$ for some $\alpha < \kappa^+$. Since the pointclasses κ^+-**Bor** and $\kappa^+\text{-}\mathbf{\Sigma}^0_\alpha$ are boldface and hereditary,

$$\kappa^+\text{-}\mathbf{\Sigma}^0_\alpha(Y) = \{A \cap Y \mid A \in \kappa^+\text{-}\mathbf{\Sigma}^0_\alpha(X)\},$$
$$\kappa^+\text{-}\mathbf{Bor}(Y) = \{A \cap Y \mid A \in \kappa^+\text{-}\mathbf{Bor}(X)\}.$$

Thus, we get $\kappa^+\text{-}\mathbf{\Sigma}^0_\alpha(Y) = \kappa^+\text{-}\mathbf{Bor}(Y)$ and the κ^+-Borel hierarchy on $({}^\kappa 2, \tau_b)$ collapses, a contradiction. □

Before stating the main theorem of the section, we need the following result from [HKSW22], which is Mycielski's theorem for ideal topologies.

Lemma 4.8. [HKSW22, Corollary 1.4] *The intersection of κ-many $\mathcal{I}$-open $\mathcal{I}$-dense subsets of ${}^\kappa 2$ contains a closed set homeomorphic to $({}^\kappa 2, \tau_b)$.*

Theorem 4.9. *The $\mathcal{I}$-Borel hierarchy on ${}^\kappa 2$ and on ${}^\kappa\kappa$ does not collapse, therefore it has length κ^+ in both spaces.*

Proof. First, we show that the $\mathcal{I}$-Borel hierarchy on ${}^\kappa 2$ has length κ^+. By Proposition 4.8 and Theorem 4.7, it is enough to show that there are κ-many $\mathcal{I}$-open $\mathcal{I}$-dense subsets of ${}^\kappa 2$.

Claim. For any $X \subseteq {}^\kappa 2$, if $|X| \leq \kappa$ then ${}^\kappa 2 \setminus X$ is $\mathcal{I}$-dense in ${}^\kappa 2$.

Proof of the Claim. Suppose, towards a contradiction, that there exists an $\mathcal{I}$-open set $U \subseteq {}^\kappa 2$ such that $U \cap ({}^\kappa 2 \setminus X) = \emptyset$. Then $U \subseteq X$, and since $|X| \leq \kappa$, also $|U| \leq \kappa$, in contradiction with Lemma 2.4. □

Let $\langle C_\alpha \mid \alpha < \kappa\rangle$ be an injective sequence of $\mathcal{I}$-closed subsets of ${}^\kappa 2$ such that $|C_\alpha| \leq \kappa$ for every $\alpha < \kappa$. For example, let $\langle \gamma_\alpha \mid \alpha < \kappa\rangle$ be an enumeration of the limit ordinals below κ and let $C_\alpha = \{x \in {}^\kappa 2 \mid |\{\beta < \kappa \mid x(\beta) = 1\}| < \gamma_\alpha\}$. Each C_α is b-closed (thus $\mathcal{I}$-closed) and of size κ. Then, $\{{}^\kappa 2 \setminus C_\alpha \mid \alpha < \kappa\}$ is a collection of κ-many distinct $\mathcal{I}$-open sets, which are also $\mathcal{I}$-dense by the Claim above.

Next, consider the $\mathcal{I}$-Borel hierarchy on ${}^\kappa\kappa$. Suppose, towards a contradiction, that there exists $\alpha < \kappa^+$ such that $\mathcal{I}\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\kappa) = \mathcal{I}\text{-}\mathbf{Bor}({}^\kappa\kappa)$. Given $\nu \in \{2, \kappa\}$, we recall that in our notation $\mathcal{I}\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\nu) = \kappa^+\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\nu, \tau_\mathcal{I})$ and $\mathcal{I}\text{-}\mathbf{Bor}({}^\kappa\kappa) = \kappa^+\text{-}\mathbf{Bor}({}^\kappa\kappa, \tau_\mathcal{I})$. Since $({}^\kappa 2, \tau_\mathcal{I})$ is a subspace of $({}^\kappa\kappa, \tau_\mathcal{I})$, and the classes $\kappa^+\text{-}\mathbf{\Sigma}^0_\alpha$ and κ^+-**Bor** are hereditary boldface:

$$\kappa^+\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa 2, \tau_\mathcal{I}) = \{A \cap {}^\kappa 2 \mid A \in \kappa^+\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\kappa, \tau_\mathcal{I})\},$$
$$\kappa^+\text{-}\mathbf{Bor}({}^\kappa 2, \tau_\mathcal{I}) = \{A \cap {}^\kappa 2 \mid A \in \kappa^+\text{-}\mathbf{Bor}({}^\kappa\kappa, \tau_\mathcal{I})\}.$$

Therefore, $\mathcal{I}\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa 2) = \kappa^+\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa 2, \tau_\mathcal{I}) = \kappa^+\text{-}\mathbf{Bor}({}^\kappa 2, \tau_\mathcal{I}) = \mathcal{I}\text{-}\mathbf{Bor}({}^\kappa 2)$. A contradiction, since we proved that the $\mathcal{I}$-Borel hierarchy on ${}^\kappa 2$ does not collapse. □

Another relevant point is whether the notion of $\mathcal{I}$-Borelness is nontrivial, that is, whether $\mathcal{I}\text{-}\mathbf{Bor}({}^\kappa\nu) \neq \mathscr{P}({}^\kappa\nu)$. In some cases, cardinality considerations are useful, for example if $\mathcal{I} = b$ and $\kappa^{<\kappa} = \kappa$ then $|\kappa^+\text{-}\mathbf{Bor}({}^\kappa\nu)| = 2^{(\nu^{<\kappa})} < |\mathscr{P}({}^\kappa\nu)|$. However, this approach does not work when we drop the assumption $2^{<\kappa} = \kappa$, or when dealing with ideal topologies. Indeed, if $\mathcal{I}$ contains an unbounded subset of κ, by Lemma 2.5(5)

$$2^{2^\kappa} = |\tau_\mathcal{I}| \leq |\mathcal{I}\text{-}\mathbf{Bor}({}^\kappa\nu)| \leq |\mathcal{P}({}^\kappa\nu)| = 2^{2^\kappa}$$

therefore $|\mathcal{I}\text{-}\mathbf{Bor}({}^\kappa\nu)| = |\tau_\mathcal{I}| = |\mathscr{P}({}^\kappa\nu)|$.

If $2^{<\kappa} = \kappa$, for any ideal $\mathcal{I}$ there exists a subset of ${}^\kappa 2$ which is not $\mathcal{I}$-Borel ([HKSW22, Observation 3.24]). If $\mathcal{I}$ is not stationarily tall, the same is true even when $2^{<\kappa} > \kappa$ (see [HKSW22, Corollary 3.23]). It remains open whether there is non-$\mathcal{I}$-Borel set if $2^{<\kappa} > \kappa$ and $\mathcal{I}$ is stationarily tall. This question also appears in [HKSW22].

In classical descriptive set theory, universal sets play a crucial role in proving the non-collapse of the Borel hierarchy on uncountable Polish spaces. When moving to the uncountable setting with $\kappa > \omega$ and considering $({}^\kappa\nu, \tau_b)$, the situation is as follows. Under the assumption $2^{<\kappa} = \kappa$, there exist ${}^\kappa 2$-universal sets for both $\kappa^+\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\nu, \tau_b)$ and $\kappa^+\text{-}\mathbf{\Pi}^0_\alpha({}^\kappa\nu, \tau_b)$ for each $1 \leq \alpha < \kappa^+$ (see, e.g., [ACRP25]). On the other hand, if the condition $2^{<\kappa} = \kappa$ fails, then neither $\kappa^+\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa 2, \tau_b)$ nor $\kappa^+\text{-}\mathbf{\Pi}^0_\alpha({}^\kappa 2, \tau_b)$ admits a ${}^\kappa 2$-universal set, for any $1 \leq \alpha < \kappa^+$ ([AMR22, Corollary 4.16]). The result below shows that when $\mathcal{I}$ contains an unbounded subset of κ, there are no universal sets for $\mathcal{I}\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\nu)$ nor $\mathcal{I}\text{-}\mathbf{\Pi}^0_\alpha({}^\kappa\nu)$, regardless of whether $2^{<\kappa} = \kappa$ holds.

Proposition 4.10. *Assume that $\mathcal{I}$ contains an unbounded subset of κ. Then, for $1 \leq \alpha < \kappa^+$, neither $\mathcal{I}\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\nu)$ nor $\mathcal{I}\text{-}\mathbf{\Pi}^0_\alpha({}^\kappa\nu)$ have a ${}^\kappa 2$-universal set.*

Proof. By Point (5) in Lemma 2.5, $\mathcal{I}\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\nu)$ has size greater than 2^{2^κ}, while $\{\mathcal{U}_y \mid y \in {}^\kappa 2\}$ has size at most 2^κ for all $\mathcal{U} \subseteq {}^\kappa 2 \times {}^\kappa\nu$. The result for $\mathcal{I}\text{-}\mathbf{\Pi}^0_\alpha({}^\kappa\nu)$ follows by taking complements. □

We continue with the closure properties of the classes in the $\mathcal{I}$-Borel hierarchy on ${}^\kappa\nu$. For the proof of the next result, we refer the reader to [ACRP25]. In that paper, the authors consider (regular Hausdorff) topological spaces of weight at most κ and work under the assumption $2^{<\kappa} = \kappa$. However, their proof remains valid without the weight restriction. Since the $\mathcal{I}$-Borel hierarchy may not satisfy the inclusion $\mathcal{I}\text{-}\mathbf{\Sigma}^0_1({}^\kappa\nu) \subseteq \mathcal{I}\text{-}\mathbf{\Sigma}^0_2({}^\kappa\nu)$, the only case requiring verification is $\alpha = 1$: by Lemma 2.5(2), the class $\mathcal{I}\text{-}\mathbf{\Sigma}^0_1({}^\kappa\nu)$ is closed under arbitrary unions and under intersections of fewer than κ sets.

Proposition 4.11. [ACRP25, Proposition 4.1] *Let $\nu \in \{2, \kappa\}$, and $2^{<\kappa} = \kappa$. Given any $1 \leq \alpha < \kappa^+$, let $\widehat{\alpha} = \kappa$ if α is a successor ordinal, and $\widehat{\alpha} = \mathrm{cof}(\alpha)$ if α is limit. Then:*

1. *$\mathcal{I}\text{-}\mathbf{\Sigma}^0_\alpha({}^\kappa\nu)$ is closed under unions of length κ and intersections of size smaller than $\widehat{\alpha}$;*
2. *$\mathcal{I}\text{-}\mathbf{\Pi}^0_\alpha({}^\kappa\nu)$is closed under intersections of length κ and unions of size smaller than $\widehat{\alpha}$;*

(3) $\mathcal{I}$-$\mathbf{\Delta}^0_\alpha({}^\kappa\nu)$ *is closed under complements and both unions and intersections of size smaller than* $\widehat{\alpha}$*, that is,* $\mathcal{I}$-$\mathbf{\Delta}^0_\alpha({}^\kappa\nu)$ *is a* $\widehat{\alpha}$*-algebra.*

We refer the reader to [ACRP25] for a detailed discussion of the optimality of the closure properties presented in 4.11.

We conclude this section by pointing out a connection with [HMV25]. When one drops the assumption $\kappa^{<\kappa} = \kappa$ (equivalently, that κ is regular and $2^{<\kappa} = \kappa$), there are two main approaches in the literature. The first maintains the condition $2^{<\kappa} = \kappa$ but allows κ to be singular, as in [AMR22, DR25, ACRP25, MRP25]; this ensures that the spaces still have weight κ. The second approach, developed in [HMV25], keeps κ regular while allowing $\kappa^{<\kappa} > \kappa$. This constitutes a different setup, as the spaces no longer have weight κ. In that context, the authors identify new connections with the model theory of uncountable structures by formulating an alternative notion of Borel sets, tailored to their setting and better suited for model-theoretic applications. They begin by defining what they call a *basic* κ*-open set*, namely a set of the form $[f] = \{x \in {}^\kappa\nu \mid f \subseteq x\}$, where $f\colon X \to \nu$ and $|X| < \kappa$. As usual, $\nu \in \{2, \kappa\}$.

Definition 4.12. The class of κ-*Borel* sets of ${}^\kappa\nu$ is the smallest class containg the basic κ-open sets and closed under complements and unions of size at most κ.

In line with Definition 4.12, one could attempt to develop the theory of Borel sets in ideal topologies using the following definition instead of the one adopted in this article (cf. page 4).

Definition 4.13. The class of **basic $\mathcal{I}$-Borel** of ${}^\kappa\nu$ is the smallest class containg the basic $\mathcal{I}$-open sets and closed under complements and unions of size at most κ.

As the next result shows, for any ideal $\mathcal{I}$ the class of basic $\mathcal{I}$-Borel sets coincides with the class defined in Definition 4.12.

Fact 4.14. For any ideal $\mathcal{I}$, the class of basic $\mathcal{I}$-Borel sets of ${}^\kappa\nu$ coincide with the class of κ-Borel sets of ${}^\kappa\nu$. As a consequence, the class of basic $\mathcal{I}$-Borel is independent of the ideal $\mathcal{I}$.

Proof. As $b \subseteq \mathcal{I}$, any basic κ-open set is a basic $\mathcal{I}$-open set, therefore every κ-Borel set is a basic $\mathcal{I}$-Borel set. For the other inclusion, it is sufficient to show that all basic $\mathcal{I}$-open sets are κ-Borel sets. As observed in Remark 4.3, for every $f \in \mathrm{Fn}_{\mathcal{I}}$, $\boldsymbol{N}_f = \bigcap_{\alpha \in D} \mathcal{N}^f_\alpha$, where $D = \mathrm{dom}(f)$ and $\mathcal{N}^f_\alpha = \{x \in {}^\kappa\nu \mid x(\alpha) = f(\alpha)\}$. Since each $\mathcal{N}^f_\alpha$ is a basic κ-open sets and $|D| \leq \kappa$, $\boldsymbol{N}_f$ is κ-Borel. □

5. The $\mathcal{I}$-Analytic sets

In this section we introduce the notion of generalized analytic set for the spaces ${}^\kappa 2$ and ${}^\kappa\kappa$ endowed with the ideal topology $\tau_{\mathcal{I}}$. However, we immediately observe that this notion is trivial, because the class under consideration actually coincides with the entire powerset (Corollary 5.3). Moreover, the same phenomenon occurs for the generalized versions of all the equivalent characterizations of analytic sets from classical descriptive set theory (Corollary 5.4). As usual, fix $\nu \in \{2, \kappa\}$.

Given $\mu \in \{2, \kappa\}$, a function $\Phi\colon {}^\kappa\nu \to {}^\kappa\mu$ is **$\mathcal{I}$-Borel** if $\Phi^{-1}(U) \in \mathcal{I}$-$\mathbf{Bor}({}^\kappa\nu)$ for every $\mathcal{I}$-open set $U \subseteq {}^\kappa\mu$. We say that a set $B \subseteq {}^\kappa\mu$ is an $\mathcal{I}$-continuous image of $A \subseteq {}^\kappa\nu$ if there is an $\mathcal{I}$-continuous function $\Phi\colon {}^\kappa\nu \to {}^\kappa\mu$ such that $\Phi(A) = B$. If Φ is $\mathcal{I}$-Borel, then we say that B is an $\mathcal{I}$-Borel image of A.

Definition 5.1. A set $A \subseteq {}^\kappa\nu$ is called **$\mathcal{I}$-analytic** ($\mathcal{I}$-$\mathbf{\Sigma}^1_1({}^\kappa\nu)$) if it is either empty or an $\mathcal{I}$-continuous image of an $\mathcal{I}$-closed subset of ${}^\kappa\kappa$.

When $\mathcal{I} = b$, $\mathcal{I}$-$\mathbf{\Sigma}^1_1({}^\kappa\nu)$ is the usual class of κ-analytic sets as in [FHK14, LS15, AMR22].

Theorem 5.2. *If $\mathcal{I}$ contains an unbounded subset of κ, then the class $\mathcal{I}$-$\mathbf{\Sigma}^1_1({}^\kappa\nu)$ is closed under unions of size at most 2^κ.*

Proof. Let $\langle A_\alpha \mid \alpha < 2^\kappa\rangle$ be a sequence of $\mathcal{I}$-$\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\nu$. For every $\alpha < 2^\kappa$, let $C_\alpha \subseteq {}^\kappa\kappa$ be $\mathcal{I}$-closed and $\Phi_\alpha\colon {}^\kappa\kappa \to {}^\kappa\nu$ be $\mathcal{I}$-continuous such that $\Phi_\alpha(C_\alpha) = A_\alpha$. Since $\mathcal{I}$ contains an unbounded set, there are $U_0, U_1 \subseteq \kappa$ unbounded such that $U_0, U_1 \in \mathcal{I}$ and $U_0 \cap U_1 = \emptyset$. Fix the bijections

$$\theta\colon {}^{U_1}\kappa \to 2^\kappa$$
$$\delta\colon U_0 \to \kappa$$

and define $\Delta\colon {}^\kappa\kappa \to {}^\kappa\kappa$ such that $\Delta(x)(\alpha) = x(\delta^{-1}(\alpha))$ for all $\alpha < \kappa$. Given $x \in {}^\kappa\kappa$, notice that for every $y \in \boldsymbol{N}_{x\restriction U_0}$, $\Delta(y) = \Delta(x)$ and, for every $z \in \boldsymbol{N}_{x\restriction U_1}$, $\theta(z \restriction U_1) = \theta(x \restriction U_1)$. Finally, let

$$\Phi\colon {}^\kappa\kappa \to {}^\kappa\nu : x \mapsto \Phi_{\theta(x\restriction U_1)}(\Delta(x)),$$

and define

$$C = \{x \in {}^\kappa\kappa \mid \Delta(x) \in C_{\theta(x\restriction U_1)}\}.$$

We claim that Φ and C witness $\bigcup_{\alpha<2^\kappa} A_\alpha \in \mathcal{I}$-$\mathbf{\Sigma}^1_1({}^\kappa\nu)$.

Notice that for every $x \in {}^\kappa\kappa$ and for every $z \in \boldsymbol{N}_{x\restriction U_0\cup U_1}$, $\Phi(z) = \Phi_{\theta(x\restriction U_1)}(\Delta(x)) = \Phi(x)$, therefore $\Phi(\boldsymbol{N}_{x\restriction U_0\cup U_1}) = \{\Phi(x)\}$.

The function Φ is $\mathcal{I}$-continuous since for every $x \in {}^\kappa\kappa$, and for every $D \in \mathcal{I}$, the set $\boldsymbol{N}_{x\restriction U_0\cup U_1}$ is $\mathcal{I}$-open (because $U_0 \cup U_1 \in \mathcal{I}$) and it satisfies $\Phi(\boldsymbol{N}_{x\restriction U_0\cup U_1}) \subseteq \boldsymbol{N}_{\Phi(x)\restriction D}$. To see that C is $\mathcal{I}$-closed we show that for every $x \notin C$, $\boldsymbol{N}_{x\restriction U_0\cup U_1} \subseteq {}^\kappa\kappa \setminus C$. This is immediate, as for every $y \in \boldsymbol{N}_{x\restriction U_0\cup U_1}$, $\Delta(y) = \Delta(x)$ and $\theta(y \restriction U_1) = \theta(x \restriction U_1)$, therefore $\Delta(y) \notin C_{\theta(y\restriction U_1)}$. The following claim concludes the proof.

Claim. $\Phi(C) = \bigcup_{\alpha<2^\kappa} A_\alpha$.

Proof of the Claim. If $y \in \bigcup_{\alpha<2^\kappa} A_\alpha$, then there is some $\alpha < 2^\kappa$ and some $x \in C_\alpha$ such that $y = \Phi_\alpha(x)$. Let $z \in {}^\kappa\kappa$ such that $\theta(z \restriction U_1) = \alpha$ and $\Delta(z) = x$. Then,

$$\Phi(z) = \Phi_{\theta(z\restriction U_1)}(\Delta(z)) = \Phi_\alpha(x) = y.$$

Finally, since $x \in C_\alpha$, $\Delta(z) \in C_{\theta(z\restriction U_1)}$. Thus $z \in C$ and $y \in \Phi(C)$.

For the other direction, let $y \in C$. From the definition of C, $\Delta(y) \in C_{\theta(y\restriction U_1)}$. Let $\alpha = \theta(y \restriction U_1)$ and $x = \Delta(y)$. Then, $x \in C_\alpha$ and $\Phi_\alpha(x) \in A_\alpha$. Therefore, $\Phi(y) = \Phi_{\theta(y\restriction U_1)}(\Delta(y)) = \Phi_\alpha(x) \in A_\alpha$. □

□

Corollary 5.3. *If $\mathcal{I}$ contains an unbounded subset of κ, then $\mathcal{I}$-$\mathbf{\Sigma}^1_1({}^\kappa\nu) = \mathscr{P}({}^\kappa\nu)$.*

The generalized Cantor and Baire spaces (equipped with the bounded topology, hence $\mathcal{I} = b$) have $2^{(2^{<\kappa})}$-many κ-analytic subsets, so the hypothesis $2^{<\kappa} = \kappa$ guarantees that the class $\mathcal{I}$-$\mathbf{\Sigma}^1_1$ is strictly smaller than the full powerset. In contrast, when $\kappa^{<\kappa} > \kappa$, [HMV25, Lemma 2.1] shows that $\mathcal{I}$-$\mathbf{\Sigma}^1_1({}^\kappa 2) = \mathscr{P}({}^\kappa 2)$ occurs even if $\mathcal{I}$ is the bounded ideal.

Since in classical descriptive set theory the notion of analytic set can be reformulated giving several equivalent definitions [Kec95, Section 14.A], the reader may wonder if the generalized counterpart of any of the other definitions may be nontrivial. This is not the case, as the next result shows.

Given a subset $A \subseteq X \times Y$ of a cartesian product, we denote its projection (on the first coordinate) by $\mathrm{p}(A) = \{x \in X \mid \exists y \in Y (x, y) \in A\}$.

Corollary 5.4. *If $\mathcal{I}$ contains an unbounded subset of κ, the following are equivalent:*

(1) $A \in \mathscr{P}({}^{\kappa}\nu)$
(2) *A is $\mathcal{I}$-analytic;*
(3) *$A = \mathrm{p}(C)$ for some $\mathcal{I}$-closed $C \subseteq {}^{\kappa}\nu \times {}^{\kappa}\kappa$;*
(4) *A is an $\mathcal{I}$-continuous image of an $\mathcal{I}$-Borel subset of ${}^{\kappa}\kappa$;*
(5) *$A = \mathrm{p}(B)$ for some $\mathcal{I}$-Borel $B \subseteq {}^{\kappa}\nu \times {}^{\kappa}\kappa$;*
(6) *A is an $\mathcal{I}$-Borel image of an $\mathcal{I}$-Borel subset of ${}^{\kappa}\kappa$.*

Proof. Trivially, (2)-(6) implies (1). Clearly (2) implies (4). (3) implies (5), since every $\mathcal{I}$-closed set is $\mathcal{I}$-Borel. (4) implies (6), since every $\mathcal{I}$-continuous function is $\mathcal{I}$-Borel. By Corollary 5.3, (1) implies (2). It remains to show that (2) implies (3). Let $A \subseteq {}^{\kappa}\nu$ be $\mathcal{I}$-analytic and let $\Phi\colon C \to {}^{\kappa}2$ be an $\mathcal{I}$-continuous surjection onto A for some $\mathcal{I}$-closed $C \subseteq {}^{\kappa}\kappa$. Since $({}^{\kappa}\kappa, \tau_{\mathcal{I}})$ is an Hausdorff space by Lemma 2.5(1), this implies that $\mathrm{Graph}(\Phi) = \{(x, y) \in C \times {}^{\kappa}\nu \mid \Phi(x) = y\}$ is $\mathcal{I}$-closed in $C \times {}^{\kappa}\nu$, hence also $\mathcal{I}$-closed in ${}^{\kappa}\kappa \times {}^{\kappa}\nu$. Clearly, $A = \mathrm{p}(\{(y, x) \in {}^{\kappa}\nu \times {}^{\kappa}\kappa \mid (x, y) \in \mathrm{Graph}(\Phi)\})$. □

Only one possible definition of $\mathcal{I}$-analytic set is missing from Corollary 5.4. In classical descriptive set theory (hence $\kappa = \omega$), it is well-known (see [Kec95, Section 14-15]) that a non-empty set $A \subseteq {}^{\omega}2$ is:

- a continuous image of ${}^{\omega}\omega$ if and only if A is analytic;
- an injective continuous image of some closed $C \subseteq {}^{\omega}\omega$ if and only if A is Borel.

Moreover, in the generalized setting $\kappa^{<\kappa} = \kappa > \omega$, the class of continuous injective images of closed subsets of $({}^{\kappa}\kappa, \tau_b)$ consistently coincide with the class of κ-analytic sets [LS15]. It is therefore natural to ask whether the generalization of at least one of these two definitions may give a nontrivial notion of $\mathcal{I}$-analytic set. The answer is negative again.

Proposition 5.5. *Assume that $\mathcal{I}$ contains an unbounded subset of κ. Then:*

(1) *Every non-empty $A \subseteq {}^{\kappa}\nu$ is an $\mathcal{I}$-continuous image of ${}^{\kappa}\kappa$.*
(2) *Every $A \subseteq {}^{\kappa}\nu$ is an injective $\mathcal{I}$-continuous image of some $\mathcal{I}$-closed $C \subseteq {}^{\kappa}\kappa$.*

Proof. Let $U \in \mathcal{I}$ be unbounded and fix a bijection $\theta : {}^{U}\kappa \to {}^{\kappa}\nu$. Then

$$\Phi : {}^{\kappa}\kappa \to {}^{\kappa}\nu : x \mapsto \theta(x \restriction U)$$

is such that for every non-empty $A \subseteq {}^{\kappa}\nu$, $\Phi^{-1}(A)$ is $\mathcal{I}$-clopen, because

$$\Phi^{-1}(A) = \bigcup \{\boldsymbol{N}_f \mid f \in {}^{U}\kappa, \theta(f) \in A\}$$

and

$${}^{\kappa}\kappa \setminus \Phi^{-1}(A) = \bigcup \{\boldsymbol{N}_f \mid f \in {}^{U}\kappa, \theta(f) \notin A\}$$

are both $\mathcal{I}$-open. In particular, $\Phi : {}^{\kappa}\kappa \to {}^{\kappa}\nu$ is $\mathcal{I}$-continuous.

To show (1), we define the surjective function $\Psi : {}^{\kappa}\kappa \to A$ by setting:

$$\Psi(x) = \begin{cases} \Phi(x) & \text{if } x \in \Phi^{-1}(A) \\ x_0 & \text{if } x \notin \Phi^{-1}(A) \end{cases}$$

for some $x_0 \in A$.

For (2), consider the set

$$C = \{x \in {}^{\kappa}\kappa \mid x(\alpha) = 0 \text{ for all } \alpha \in \kappa \setminus U, \theta(x \restriction U) \in A\}.$$

Since $|\kappa \setminus U| \leq \kappa$ and $\{x \in {}^\kappa\kappa \mid x(\alpha) \neq 0\} \in \kappa^+\text{-}\mathbf{\Delta}^0_1 \subseteq \mathcal{I}\text{-}\mathbf{\Delta}^0_1$,

$$ {}^\kappa\kappa \setminus C = \bigcup_{\alpha \in \kappa \setminus U} \{x \in {}^\kappa\kappa \mid x(\alpha) \neq 0\} \cup \bigcup \{\boldsymbol{N}_f \mid \theta(f) \notin A\} $$

is $\mathcal{I}$-open. Thus C is $\mathcal{I}$-closed. Then $\Phi \upharpoonright C$ is injective by construction (since θ is bijective) and it is onto A. □

6. The Approximation Lemma

The Approximation Lemma has been one of the main techniques for constructing continuous reductions between equivalence relations in the generalized Baire space (i.e. endowed with the bounded topology under the assumption $\kappa^{<\kappa} = \kappa$). In [FMR21] this method was also extensively used to study reductions between different equivalence relations under some reflection principles. We refer the reader to [FMR21] for all undefined notions in this section.

One of the main applications of the Approximation Lemma in the bounded topology is the characterization of some reductions, for instance the ones in Example 6.4. In this framework, if $S, X \subseteq \kappa$ are disjoint stationary sets, then the following are equivalent:

(1) $=^\kappa_S$ is Lipschitz reducible to $=^\kappa_X$;
(2) $=^\kappa_S$ is X-recursive reducible to $=^\kappa_X$;
(3) $=^\kappa_S$ has an X-approximation with the club filter[6].

All these considerations motivate the study of the Approximation Lemma for ideal topologies. In this section we study the Approximation Lemma for the space $({}^\kappa\kappa, \tau_\mathcal{I})$. Since we consider only ideals that have the approximation property (Definition 2.1), throughout this section we assume that $\mathcal{I}$ has a basis of size κ.

Definition 6.1. Let $\langle B_\alpha \mid \alpha < \kappa\rangle$ be an approximation sequence of $\mathcal{I}$, and $A \subseteq \kappa$ an unbounded set. We say that a function $f : {}^\kappa\kappa \to {}^\kappa\kappa$ is **A-recursive over $\langle B_\alpha\rangle_{\alpha<\kappa}$** if there exists a monotone function

$$ H : \mathrm{Fn}_\mathcal{I} \to \mathrm{Fn}_\mathcal{I} $$

such that for every $\alpha \in A$ and for every $x \in {}^\kappa\kappa$,

$$ f(x)(\theta) = H(x \upharpoonright B_\alpha)(\theta), $$

for every $\theta \in B_{\alpha'} \cup (A \cap \alpha')$, where $\alpha' = min(A \backslash (\alpha + 1))$.

Let $D \in \mathcal{I}$, since $\langle B_\alpha \mid \alpha < \kappa\rangle$ is an approximation sequence of $\mathcal{I}$, there is $\alpha < \kappa$ such that $D \subseteq B_\alpha$. Since $\alpha < \alpha' = min(A \backslash (\alpha + 1))$, we have $B_\alpha \subseteq B_{\alpha'} \subseteq \mathrm{dom}(H(x \upharpoonright B_\alpha))$, thus $D \subseteq \mathrm{dom}(H(x \upharpoonright B_\alpha))$. Therefore, if H witnesses that f is A-recursive over $\langle B_\alpha \mid \alpha < \kappa\rangle$, then H is continuous as in Definition 3.1. In particular $H^* = f$, thus f is $\mathcal{I}$-continuous by Proposition 3.2.

Fact 6.2. Let $\langle B_\alpha \mid \alpha < \kappa\rangle$ be an approximation sequence of $\mathcal{I}$, and $A \subseteq \kappa$ an unbounded set. If f is A-recursive over $\langle B_\alpha \mid \alpha < \kappa\rangle$, then f is $\mathcal{I}$-continuous.

Let E_1 and E_2 be equivalence relations on ${}^\kappa\kappa$. Let $A \subseteq \kappa$ be an unbounded set. We say that E_1 **is A-recursive reducible to** E_2 if there are an approximation

[6]See Example 6.4 for the definition of $=^\kappa_S$, and Definition 6.5 for the definition of X-approximation.

sequence $\langle B_\alpha \mid \alpha < \kappa\rangle$ of $\mathcal{I}$, and a function $f\colon {}^\kappa\kappa \to {}^\kappa\kappa$ A-recursive over $\langle B_\alpha \mid \alpha < \kappa\rangle$ that satisfies

$$(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2.$$

We call f an **A-recursive reduction** of E_1 to E_2, and when such reduction exists we write $E_1 \hookrightarrow_{R(A)} E_2$. Clearly, $E_1 \hookrightarrow_{R(A)} E_2$ implies that an $\mathcal{I}$-continuous ($\mathcal{I}$-Borel) reduction exists. Thus an A-recursive reduction is stronger than an $\mathcal{I}$-continuous reduction.

Definition 6.3. Let $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ be a filter. We define the following equivalence relation $\sim^\kappa_\mathcal{F}$ on the space ${}^\kappa\kappa$:

$$x \sim^\kappa_\mathcal{F} y \Leftrightarrow \exists W \in \mathcal{F}\ \left(W \subseteq \{\alpha < \kappa \mid x(\alpha) = y(\alpha)\}\right).$$

Given a filter $\mathcal{F}$, we denote by $\mathcal{F}^+$ the set $\{X \subseteq \kappa \mid \forall Y \in \mathcal{F}(X \cap Y \neq \emptyset)\}$. For every $A \in \mathcal{F}^+$, we define the filter $\mathcal{F} \restriction A$ as the filter generated by $\{X \cap A \mid X \in \mathcal{F}\}$.

Example 6.4. Many well-known equivalence relation are induced by a filter like in Definition 6.3. The following are some examples.

- The identity relation id is the equivalence relation induced by the filter $\{\kappa\}$.
- For $X \subseteq \kappa$, the equivalence modulo X, denoted by X_κ, is the equivalence relation induced by the filter $\{\kappa\} \restriction X = \{Y \subseteq \kappa \mid X \subseteq Y\}$.
- $E_0^{<\kappa,\kappa} = \{(x, y) \in {}^\kappa\kappa \times {}^\kappa\kappa \mid \exists\alpha < \kappa\forall\beta > \alpha\ (x(\beta) = y(\beta))\}$ is the equivalence relation induced by the filter generated by $\{\kappa \setminus \alpha \mid \alpha < \kappa\}$.
- The equivalence modulo non-stationary set $=^\kappa_{\mathrm{CUB}}$ is the equivalence relation induced by the club filter CUB.
- For S be a stationary subset of κ, $=^\kappa_S$ is the equivalence relation induced by the filter CUB $\restriction S$.

Given an approximation sequence $\langle B_\alpha\rangle_{\alpha<\kappa}$ of $\mathcal{I}$, we define the search function $s : \kappa \to \kappa$ as follows:

$$s(\alpha) = \begin{cases} \beta & \text{if } \alpha \in B_{\beta+1} \backslash B_\beta \text{ and } \beta < \alpha, \\ \alpha & \text{otherwise.} \end{cases}$$

Note that s is well-defined because $\langle B_\alpha\rangle_{\alpha<\kappa}$ is an strictly increasing continuous sequence.

Definition 6.5. Let $\langle B_\alpha \mid \alpha < \kappa\rangle$ be an approximation sequence of $\mathcal{I}$, and let $\mathcal{H}$ be a filter over κ with basis $\mathcal{B}$. Let $A \in \mathcal{H}^+$ be an unbounded set and let E be an equivalence relation on ${}^\kappa\kappa$. We say that E has an **A-approximation over** $\langle B_\alpha \mid \alpha < \kappa\rangle$ if there is a sequence $\langle E_\alpha \mid \alpha < \kappa\rangle$ with $E_\alpha \subseteq {}^{B_{s(\alpha)}}\kappa \times {}^{B_{s(\alpha)}}\kappa$ such that the following conditions hold:

(1) For every $\alpha < \kappa$, E_α is an equivalence relation.
(2) For every $x, y \in {}^\kappa\kappa$, if xEy then there exists $D \in \mathcal{B}$ such that for every $\alpha \in D$, $x \restriction B_{s(\alpha)}\ E_\alpha\ y \restriction B_{s(\alpha)}$.
(3) For every $x, y \in {}^\kappa\kappa$, if $\neg(xEy)$ then there exists $A' \in \mathcal{H}^+$, $A' \subseteq A$, such that for every $\alpha \in A'$, $\neg(x \restriction B_{s(\alpha)}\ E_\alpha\ y \restriction B_{s(\alpha)})$.

An A-approximation is said to be **precise** if for every $\alpha < \kappa$, E_α has at most κ-many classes. Clearly, the assumption $\kappa^{<\kappa} = \kappa$ implies that every A-approximation is precise. We say that E has an (precise) A-approximation if there is an approximation sequence $\langle B_\alpha \mid \alpha < \kappa\rangle$ of $\mathcal{I}$, such that E has an (precise) A-approximation over $\langle B_\alpha \mid \alpha < \kappa\rangle$.

Lemma 6.6 (Approximation Lemma)**.** *Let $\mathcal{H}$ be a filter over κ with basis $\mathcal{B}$. Let $A \in \mathcal{H}^+$ be an unbounded set and let E an equivalence relation on ${}^\kappa\kappa$. Then, the following are equivalent:*

(1) *E has a precise A-approximation;*
(2) *$E \hookrightarrow_{R(A)} \sim^\kappa_\mathcal{F}$, where $\mathcal{F} = \mathcal{H} \restriction A$.*

Proof. (1)$\Rightarrow$ (2). Let $\langle B_\alpha \mid \alpha < \kappa\rangle$ be an approximation sequence of $\mathcal{I}$ such that there is $\langle E_\alpha \mid \alpha < \kappa\rangle$ a precise A-approximation over $\langle B_\alpha \mid \alpha < \kappa\rangle$. For every $\alpha < \kappa$, let $\langle x^\alpha_i \mid 0 < i < \kappa\rangle$ be an enumeration of the equivalence classe E_α. Let us define $F : {}^\kappa\kappa \to {}^\kappa\kappa$ as follows:

$$F(x)(\alpha) = \begin{cases} i & \text{if } \alpha \in A \text{ and } x \restriction B_{s(\alpha)} \in x^\alpha_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that $x\ E\ y$ if and only if $F(x)\ \sim^\kappa_\mathcal{F}\ F(y)$.

Claim. $x\ E\ y$ implies $F(x)\ \sim^\kappa_\mathcal{F}\ F(y)$.

Proof of the Claim. Suppose $x, y \in {}^\kappa\kappa$ are such that $x\ E\ y$. Since $\langle E_\alpha \mid \alpha < \kappa\rangle$ is an A-approximation over $\langle B_\alpha \mid \alpha < \kappa\rangle$, by Definition 6.5(2) there exists $D \in \mathcal{B}$ such that for every $\alpha \in D$,

$$x \restriction B_{s(\alpha)}\ E_\alpha\ y \restriction B_{s(\alpha)}.$$

So, for all $\alpha \in D \cap A$, $F(x)(\alpha) = F(y)(\alpha)$. Since $D \cap A \in \mathcal{H} \restriction A = \mathcal{F}$, we get $F(x)\ \sim^\kappa_\mathcal{F}\ F(y)$. □

Claim. $\neg(x\ E\ y)$ implies $\neg(F(x)\ \sim^\kappa_\mathcal{F}\ F(y))$.

Proof of the Claim. Suppose $x, y \in {}^\kappa\kappa$ are such that $\neg(x\ E\ y)$. Since $\langle E_\alpha \mid \alpha < \kappa\rangle$ is an A-approximation over $\langle B_\alpha \mid \alpha < \kappa\rangle$, by Definition 6.5(3), there is $A' \subseteq A$ in $\mathcal{H}^+$, such that for all $\alpha \in A'$,

$$\neg(x \restriction B_{s(\alpha)}\ E_\alpha\ y \restriction B_{s(\alpha)}).$$

So, for all $\alpha \in A'$, $F(x)(\alpha) \neq F(y)(\alpha)$. Since $A' \in \mathcal{H}^+$, for all $X \in \mathcal{H}$, $X \cap A' \neq \emptyset$. Thus, for every $X \in \mathcal{H}$, $(X \cap A) \cap A' \neq \emptyset$. So $A' \in (\mathcal{H} \restriction A)^+ = \mathcal{F}^+$, and we conclude that $\neg(F(x)\ \sim^\kappa_\mathcal{F}\ F(y))$. □

Claim. F is A-recursive over $\langle B_\alpha \mid \alpha < \kappa\rangle$.

Proof of the Claim. Define $H : \mathrm{Fn}_\mathcal{I} \to \mathrm{Fn}_\mathcal{I}$ as follows:

$$H(x \restriction D) = F(x) \restriction B_\alpha \cup (A \cap \alpha)$$

where α is the smallest element of A such that $D \subsetneq B_\alpha$. Notice that if $D = B_\alpha, \alpha \in A$, then $H(x \restriction D) = F(x) \restriction B_{\alpha'} \cup (A \cap \alpha')$ where $\alpha' = min(A \backslash (\alpha + 1))$.

To see that H is well defined is sufficient to check the case $\mathrm{dom}(x) = B_\alpha$ for some $\alpha \in A$. Let $x, y \in \kappa^\kappa$ be such that $x \restriction B_\alpha = y \restriction B_\alpha$. Let $\alpha' = min(A \backslash (\alpha + 1))$. Clearly, for every $\beta \in B_{\alpha'} \backslash A$, $F(x)(\beta) = 0 = F(y)(\beta)$, therefore $F(x) \restriction B_{\alpha'}\ (\beta) = 0 = F(y) \restriction B_{\alpha'}\ (\beta)$ for all $\beta \in B_{\alpha'} \backslash A$. On the other hand, let $\beta \in (B_{\alpha'} \cap A) \cup (A \cap \alpha')$. From the definition of F, $F(x)(\beta) = i$ and $F(y)(\beta) = j$, where $x \restriction B_{s(\beta)} \in x^\beta_i$ and $y \restriction B_{s(\beta)} \in x^\beta_j$. Since $s(\beta) \leq \beta$ and $\beta \in (B_{\alpha'} \cap A) \cup (A \cap \alpha')$, $s(\beta) \leq \alpha$. Thus $x \restriction B_{s(\beta)} = y \restriction B_{s(\beta)}$, $x^\beta_i = x^\beta_j$, and $i = j$. So $F(x)(\beta) = F(y)(\beta)$ for all $\beta \in (B_{\alpha'} \cap A) \cup (A \cap \alpha')$. We conclude that $F(x) \restriction B_{\alpha'} \cup (A \cap \alpha') = F(y) \restriction B_{\alpha'} \cup (A \cap \alpha')$, hence $H(x \restriction B_\alpha) = H(y \restriction B_\alpha)$ and H is well defined.

Finally, F is A-recursive by definition of H. □

(2)⇒ (1). Let $\langle B_\alpha \mid \alpha < \kappa\rangle$ be an approximation sequence of $\mathcal{I}$ and $f : {}^\kappa\kappa \to {}^\kappa\kappa$ be an A-recursive function over $\langle B_\alpha \mid \alpha < \kappa\rangle$ witnessing $E \hookrightarrow_{R(A)} \sim^\kappa_{\mathcal{F}}$. For every $\alpha < \kappa$, define $E_\alpha \subseteq {}^{B_{s(\alpha)}}\kappa \times {}^{B_{s(\alpha)}}\kappa$ as follows:

- if $\alpha \notin A$, let $E_\alpha = {}^{B_{s(\alpha)}}\kappa \times {}^{B_{s(\alpha)}}\kappa$;
- if $\alpha \in A$, let E_α be such that $x\ E_\alpha\ y$ if and only if $f(x^0)(\alpha) = f(y^0)(\alpha)$.

We now show that $\langle E_\alpha \mid \alpha < \kappa\rangle$ is a precise A-approximation over $\langle B_\alpha \mid \alpha < \kappa\rangle$. It is clear that E_α is an equivalence relation for every $\alpha < \kappa$, with at most κ many classes. To see that $\langle E_\alpha \mid \alpha < \kappa\rangle$ satisfies Definition 6.5(2), let $x, y \in {}^\kappa\kappa$ such that xEy. Since f is an A-reduction from E to $\sim^\kappa_{\mathcal{F}}$, $f(x) \sim^\kappa_{\mathcal{F}} f(y)$. Thus there is $W \in \mathcal{F}$ such that $W \subseteq \{\alpha < \kappa \mid f(x)(\alpha) = f(y)(\alpha)\}$. Since $\mathcal{F} = \mathcal{H} \restriction A$, there is $D \in \mathcal{B}$ such that $D \cap A \subseteq \{\alpha < \kappa \mid f(x)(\alpha) = f(y)(\alpha)\}$. On the other hand f is A-recursive over $\langle B_\alpha \mid \alpha < \kappa\rangle$, there is $H : \mathrm{Fn}_{\mathcal{I}} \to \mathrm{Fn}_{\mathcal{I}}$ such that for all $\alpha \in A$,

$$f((x \restriction B_\alpha)^0)(\alpha) = H((x \restriction B_\alpha)^0 \restriction B_\alpha)(\alpha) = H(x \restriction B_\alpha)(\alpha) = f(x)(\alpha)$$

and

$$f(y)(\alpha) = f((y \restriction B_\alpha)^0)(\alpha).$$

From the definition of s, we know that $s(\alpha) \leq \alpha$, thus

$$f(x)(\alpha) = f((x \restriction B_{s(\alpha)})^0)(\alpha) \text{ and } f(y)(\alpha) = f((y \restriction B_{s(\alpha)})^0)(\alpha).$$

Since for all $\alpha \in D\cap A$, $f(x)(\alpha) = f(y)(\alpha)$, $f((x \restriction B_{s(\alpha)})^0)(\alpha) = f((y \restriction B_{s(\alpha)})^0)(\alpha)$ holds for all $\alpha \in D \cap A$. We conclude that for all $\alpha \in D \cap A$, $x \restriction B_{s(\alpha)}\ E_\alpha\ y \restriction B_{s(\alpha)}$.

We conclude by showing that $\langle E_\alpha \mid \alpha < \kappa\rangle$ satisfies Definition 6.5(3). Let $x, y \in {}^\kappa\kappa$ such that $\neg(x\ E\ y)$. There is $Y \in \mathcal{F}^+$ such that for all $\alpha \in Y$, $f(x)(\alpha) \neq f(y)(\alpha)$. Since $\mathcal{F} = \mathcal{H} \restriction A$, $A \cap Y \in \mathcal{F}^+$. Let us denote $A \cap Y$ by A'. Therefore, $A' \subseteq A$ is such that for all $\alpha \in A'$, $f(x)(\alpha) \neq f(y)(\alpha)$. Using a similar argument as above, we conclude that $f((x \restriction B_{s(\alpha)})^0)(\alpha) \neq f((y \restriction B_{s(\alpha)})^0)(\alpha)$ holds for all $\alpha \in A'$. We conclude that for all $\alpha \in A'$, $\neg(x \restriction B_{s(\alpha)}\ E_\alpha\ y \restriction B_{s(\alpha)})$. □

Corollary 6.7. *Let $\mathcal{H} = \{\kappa\}$, E an equivalence relation on ${}^\kappa\kappa$ and let $X \subseteq \kappa$ be an unbounded set. Then, E has a precise X-approximation if and only if $E \hookrightarrow_{R(X)} X_\kappa$. In particular, E has a precise κ-approximation if and only if $E \hookrightarrow_{R(\kappa)} \mathrm{id}$.*

Corollary 6.8. *Let $\mathcal{H} = \mathrm{CUB}$, E an equivalence relation on ${}^\kappa\kappa$, and let $S \subseteq \kappa$ be a stationary set. Then, E has a precise S-approximation if and only if $E \hookrightarrow_{R(S)} =^\kappa_S$. In particular, E has a precise κ-approximation if and only if $E \hookrightarrow_{R(\kappa)} =^\kappa_{\mathrm{CUB}}$.*

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