On Σ_1^1-completeness of quasi-orders on κ^κ

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Abstract

We prove under $V = L$ that the inclusion modulo the non-stationary ideal is a Σ_1^1-complete quasi-order in the generalized Borel-reducibility hierarchy ($\kappa > \omega$). This improvement to known results in L has many new consequences concerning the Σ_1^1-completeness of quasi-orders and equivalence relations such as the embeddability of dense linear orders as well as the equivalence modulo various versions of the non-stationary ideal. This serves as a partial or complete answer to several open problems stated in literature. Additionally the theorem is applied to prove a dichotomy in L: If the isomorphism of a countable first-order theory (not necessarily complete) is not Δ_1^1, then it is Σ_1^1-complete.

We also study the case $V \neq L$ and prove Σ_1^1-completeness results for weakly ineffable and weakly compact κ.

Acknowledgments

The second author wishes to thank the Academy of Finland for the support through its grant number 285203 as well as both Aalto University and University of Helsinki for providing suitable research environment during the academic year 2018-2019.

The third author wishes to thank the University of Helsinki and its Doctoral Programme in Mathematics and Statistics (DOMAST) for support. He also thank Bar-Ilan University and Assaf Rinot for the support provided by the European Research Council through its grant agreement ERC-2018-StG 802756.

1 Introduction

We work in the setting of generalized descriptive set theory [FHK], GDST for short. The spaces $\kappa^\kappa = \{f : \kappa \to \kappa\}$ and $2^\kappa = \{f : \kappa \to 2\}$ are equipped with the bounded topology where the basic open sets are of the form $\{\eta \in \kappa^\kappa \mid \eta \supset p\}$, $p \in \kappa^{<\kappa}$. Borel sets are generated by κ-long unions and intersection of basic open sets. Notions of Borel-reducibility between equivalence relations and quasi-orders as well as Wadge-reducibility between sets are generalized accordingly. A set is Σ^1_1 if it is the projection of a Borel set, see next section for more detailed definitions.

In [FHK] a Lemma was introduced (a version of the Lemma and a detailed proof can be found in [HK, Lemma 1.9 & Remark 1.10]) saying that if $V = L$, then any Σ^1_1 subset of κ^κ can be Wadge-reduced to

$$\mathrm{CLUB} = \{\eta \in 2^\kappa \mid \eta^{-1}\{1\} \text{ contains a } \mu\text{-club}\}, \quad \mu < \kappa \text{ regular},$$

where "μ-club" is short for unbounded set closed under increasing sequences of length μ. In [FHK] this was used to show that if $V = L$, then $\Sigma^1_1 = \text{Borel}^*$. In [HK] the Wadge-reducibility result was strengthened by the first two authors of the present paper. It was shown (still in L) that every Σ^1_1-equivalence relation is Borel-reducible to the following equivalence relation on κ^κ:

$$E^\kappa_\mu = \{(\eta, \xi) \in (\kappa^\kappa)^2 \mid \{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\} \text{ contains a } \mu\text{-club}\}. \tag{1.1}$$

We say that E^κ_μ is Σ^1_1-*complete.*

This result was important, but we would have wanted to prove a stronger result, namely that the same equivalence relation on 2^κ is Σ^1_1-complete:

$$E^2_\mu = \{(\eta, \xi) \in (2^\kappa)^2 \mid \{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\} \text{ contains a } \mu\text{-club}\}. \tag{1.2}$$

The reason for this was that we knew many more equivalence relations to which E^2_μ can be Borel-reduced than equivalence relations to which E^κ_μ can be Borel-reduced. The corollaries of (1.1) and (1.2) were explored in [FHK, HK, Mor]. In particular, the question "Is E^κ_μ Borel-reducible to E^2_μ?" that was stated in [FHK15, Q. 15] and re-stated in [KLLS, Q. 3.46] was open (and it is still open in the general case). Of course if E^2_μ is Σ^1_1-complete the answer to this question is positive and in the present paper we show that this is the case in L (Theorem 4.2) by first proving a corresponding result for quasi-orders (Theorem 3.1). Borel-reducibility between quasi-orders is a natural generalization of reducibility between equivalence relations (see Section 2 for precise definitions).

We then prove a range of new results which are all consequences of Theorem 3.1. One of these is our main result: If $V = L$, then the isomorphism relation of *any* countable first-order theory (not necessarily complete) is either Δ^1_1 or Σ^1_1-complete. A closely related classification problem in the generalized Baire space was studied in [HKM], the so-called "Borel-reducibility counterpart of the Shelah's main gap theorem". The other results of this paper are partial answers to [Mot, Q.'s 11.3 and 11.4] (which are re-stated as [KLLS, Q's 3.49 and 3.50]), [FHK15, Q. 15] and a complete answer to [KLLS, Q. 3.47].

These questions ask about the (consistency of) reducibility between relations of the form E^λ_μ, $\lambda \in \{2, \kappa\}$, $\mu \in \mathrm{reg}(\kappa)$, quasi-orders arising as subset relations modulo certain ideals (like the μ-non-stationary ideal), quasi-orders of embeddability between linear orders as well as various isomorphism relations. In particular, [Mot, Q. 11.4] asks whether the embeddability of dense linear orders $\sqsubseteq_{\mathrm{DLO}}$ is a Σ^1_1-complete quasi-order for weakly compact κ. From those results that are described above it follows that $\sqsubseteq_{\mathrm{DLO}}$ is Σ^1_1-complete in L for all κ that are not successors of an ω-cofinal cardinal (Theorem 4.4). Since $\sqsubseteq_{\mathrm{DLO}}$ is Borel-reducible to $\sqsubseteq_G$, the embeddability of graphs, this quasi-order is also Σ^1_1-complete in this scenario. In Section 5.1 we extend this to weakly ineffable cardinals (without the assumption $V = L$). Thus the only case in which [Mot, Q. 11.4] is still open is the case when $V \neq L$ and κ is a weakly compact cardinal which is not weakly ineffable. In Section 5.2 we prove that the isomorphism of DLO, $\cong_{\mathrm{DLO}}$, on κ weakly compact is Σ^1_1-complete (here again, we do not assume $V = L$) and this implies the same for $\cong_G$, the isomorphism of graphs. The existence of Σ^1_1-complete isomorphism relations has been previously known to hold in L [HK18]. It is still unknown whether there exists a model of ZFC and $\kappa > \omega$ on which no isomorphism relation on models of size κ is Σ^1_1-complete (a stark contrast to the case $\kappa = \omega$ where the isomorphism relation on any class of countable structures is induced by a Polish group action and therefore not Σ^1_1-complete [KL]). Given the present situation such a counterexample will have to satisfy both $V \neq L$ and κ is not weakly compact.

2 Preliminaries and Definitions

In this section we define the notions and concepts we work with. Throughout this article we assume that κ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$ which is a standard assumption in the GDST. In this paper, however, this assumption is mostly redundant, because we work either with strongly inaccessible κ or under the assumption $V = L$. For sets X and Y denote by X^Y the set of all functions from Y to X. For ordinal α denote by $X^{<\alpha}$ the set of all functions from any $\beta < \alpha$ to X. We work with the generalized Baire and Cantor spaces associated with κ these being κ^κ and 2^κ respectively, where $2 = \{0, 1\}$. The generalized Baire space κ^κ is equipped with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$\{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set. The open sets are of the form $\bigcup X$ where X is a collection of basic open sets. The collection of κ-*Borel* subsets of κ^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ. A κ-*Borel set* is any element of this collection. In this paper we do not consider any other kind of Borel sets, so we always omit the prefix "κ-". The subspace $2^\kappa \subset \kappa^\kappa$ (the generalized Cantor space) is equipped with the subspace topology. We will also work in the subspaces of the form Mod^κ_T which are sets of codes for models with domain κ of a first-order countable theory T. Special cases include Mod^κ_G and $\mathrm{Mod}^\kappa_{\mathrm{DLO}}$ for graphs and dense linear orders respectively. These are Borel subspaces of 2^κ. This enables us to view the quasi-order of embeddability of models, say $\sqsubseteq_{\mathrm{DLO}}$, as a quasi-order on 2^κ. In order to precisely define this, we have to introduce some notions.

The following is a standard way to code structures with domain κ by elements of κ^κ (see e.g. [FHK]). Suppose $\mathcal{L} = \{P_n \mid n < \omega\}$ is a countable relational vocabulary.

Definition 2.1. Fix a bijection $\pi\colon \kappa^{<\omega} \to \kappa$. For every $\eta \in 2^\kappa$ define the $\mathcal{L}$-structure $\mathcal{A}_\eta$ with domain κ as follows: For every relation P_m with arity n, every tuple $(a_1, a_2, \dots, a_n)$ in κ^n satisfies

$$(a_1, \dots, a_n) \in P_m^{\mathcal{A}_\eta} \iff \eta(\pi(m, a_1, \dots, a_n)) = 1.$$

Note that for every $\mathcal{L}$-structure $\mathcal{A}$ with $\mathrm{dom}(\mathcal{A}) = \kappa$ there exists $\eta \in 2^\kappa$ with $\mathcal{A} = \mathcal{A}_\eta$. It is clear how this coding can be modified for a finite vocabulary. For club many $\alpha < \kappa$ we can also code the $\mathcal{L}$-structures with domain α:

Definition 2.2. Denote by C_π the club $\{\alpha<\kappa \mid \pi[\alpha^{<\omega}]\subseteq\alpha\}$. For every $\eta\in 2^\kappa$ and every $\alpha\in C_\pi$ define the structure $\mathcal{A}_{\eta\restriction\alpha}$ with domain α as follows: For every relation P_m with arity n, every tuple $(a_1,a_2,\dots,a_n)$ in α^n satisfies

$$(a_1,a_2,\dots,a_n)\in P_m^{\mathcal{A}_{\eta\restriction\alpha}} \iff (\eta\restriction\alpha)(\pi(m,a_1,a_2,\dots,a_n))=1.$$

Note that for every $\alpha\in C_\pi$ and every $\eta\in 2^\kappa$ the structures $\mathcal{A}_{\eta\restriction\alpha}$ and $\mathcal{A}_\eta\restriction\alpha$ are the same.

Let us denote by Mod^κ_T the subset of 2^κ consisting of those elements that code the models of a first-order countable theory T (not necessarily complete). Abbreviate first-order countable theory as FOCT from now on. We will be interested in particular in $T=G$, the theory of graphs (symmetric and irreflexive) and $T=\mathrm{DLO}$, the theory of dense linear orders without end-points. We consider Mod^κ_T as a topological space endowed with the subspace topology.

We can now define some central relations for this paper. A *quasi-order* is a transitive and reflexive relation.

Definition 2.3 (Relations)**.** We will use the following relations.

Isomorphism For a FOCT T, define

$$\cong^\kappa_T=\cong_T=\{(\eta,\xi)\in 2^\kappa\times 2^\kappa \mid \eta,\xi\in\mathrm{Mod}^\kappa_T, \mathcal{A}_\eta\cong\mathcal{A}_\xi \text{ or } \eta,\xi\notin\mathrm{Mod}^\kappa_T\}.$$

Embeddability For a FOCT T, define the quasi-order

$$\sqsubseteq^\kappa_T=\sqsubseteq_T=\{(\eta,\xi)\in(\mathrm{Mod}^\kappa_T)^2 \mid \mathcal{A}_\eta \text{ is embeddable into } \mathcal{A}_\xi\}$$

Thus, for example $\sqsubseteq_G$ is the embeddability of graphs and $\sqsubseteq_{\mathrm{DLO}}$ is the embeddability of dense linear orders.

Bi-embeddability For a FOCT T and $\eta,\xi\in\mathrm{Mod}^\kappa_T$, let

$$\eta\approx_T\xi \iff \eta\sqsubseteq_T\xi\wedge\xi\sqsubseteq_T\eta.$$

Inclusion mod NS For $\eta,\xi\in 2^\kappa$ and a stationary $S\subseteq\kappa$, we write $\eta\sqsubseteq^S\xi$ if $(\eta^{-1}\{1\}\backslash\xi^{-1}\{1\})\cap S$ is non-stationary.

Equivalence mod NS For every stationary $S\subseteq\kappa$ and $\lambda\in\{2,\kappa\}$, we define E^λ_S as the relation

$$E^\lambda_S=\{(\eta,\xi)\in\lambda^\kappa\times\lambda^\kappa \mid \{\alpha<\kappa\mid\eta(\alpha)\neq\xi(\alpha)\}\cap S \text{ is not stationary}\}.$$

Note that $\eta E^2_S\xi$ if and only if $\eta\sqsubseteq^S\xi\wedge\xi\sqsubseteq^S\eta$.

If S is the set of all μ-cofinal ordinals, denote $E^\lambda_S = E^\lambda_\mu$ and $\sqsubseteq^S = \sqsubseteq^\mu$. If S is the set of all regular cardinals below κ, denote $S = \operatorname{reg}(\kappa) = \operatorname{reg}$ in which case $E^\lambda_S = E^\lambda_{\operatorname{reg}}$ and $\sqsubseteq^S = \sqsubseteq^{\operatorname{reg}}$. If $S = \kappa$, write $E^\lambda_S = E^\lambda_{\mathrm{NS}}$ and $\sqsubseteq^S = \sqsubseteq^{\mathrm{NS}}$

A quasi-order Q on (a Borel set) $X \subseteq \kappa^\kappa$ is Σ^1_1, if $Q \subseteq X^2$ is the projection of a closed set in $X^2 \times \kappa^\kappa$ (X is equipped with subspace topology and $X^2 \times \kappa^\kappa$ with the product topology). All quasi-orders of Definition 2.3 (note that equivalence relations are quasi-orders) are Σ^1_1.

Suppose $X, Y \subseteq \kappa^\kappa$ are Borel. A function $f\colon X \to Y$ is *Borel*, if for every open set $A \subseteq Y$ the inverse image $f^{-1}[A]$ is a Borel subset of X with respect to the induced Borel structure on X and Y.

If Q_1 and Q_2 are quasi-orders respectively on X and Y, then we say that Q_1 is *Borel-reducible* to Q_2, if there exists a Borel map $f\colon X \to Y$ such that for all $x_1, x_2 \in X$ we have $x_1 Q_1 x_2 \iff f(x_1) Q_2 f(x_2)$ and this is also denoted by $Q_1 \leqslant_B Q_2$. If f is continuous (inverse image of an open set is open), then we say that Q_1 is *continuously reducible* to Q_2. Note that equivalence relations are quasi-orders, so this gives naturally a notion of reducibility for them as well. We will interchangeably use the notations xEy and $(x, y) \in E$ if E is a binary relation, because we consider it as a set of pairs. Sometimes one notation is clearer than the other.

Note that if we define $F\colon 2^\kappa \to 2^\kappa$ by

$$F(\eta)(\alpha) = \begin{cases} \eta(\alpha) & \text{if } \alpha \in S \\ 1 & \text{otherwise} \end{cases}$$

for a fixed $S \subseteq \kappa$, we obtain:

Fact 2.4. For all stationary $S \subseteq S'$ we have $\sqsubseteq^S \leqslant_B \sqsubseteq^{S'}$. □

A quasi-order is Σ^1_1-*complete*, if every Σ^1_1 quasi-order is Borel-reducible to it. An equivalence relation is Σ^1_1-*complete* if every Σ^1_1 equivalence relation is Borel-reducible to it.

A Borel equivalence relation E on a Borel subspace $X \subseteq 2^\kappa$ can be extended to 2^κ by declaring all other elements equivalent to each other, but not equivalent to any of the elements in X. Similarly a quasi-order $\sqsubseteq$ on $X \subseteq 2^\kappa$ can be trivially extended to the whole space 2^κ. If the original equivalence relation or quasi-order was Σ^1_1-complete, then so are the extensions.

3 Σ_1^1–completeness of $\sqsubseteq^S$ in L

This section is devoted to proving Theorem 3.1. In Section 4 a range of corollaries will be proved.

Theorem 3.1. $(V = L, \kappa > \omega)$ *The quasi-order $\sqsubseteq^\mu$ is Σ_1^1–complete, for every regular $\mu < \kappa$.*

As mentioned in Introduction, this is an improvement to a theorem in [HK] which says that E^κ_μ is Σ_1^1-complete.

Definition 3.2. We will need a version of the diamond principle. Denote by $\mathbb{ON}$ the class of all ordinals.

- Let us define a class function $F_\diamond \colon \mathbb{ON} \to L$. For all α, $F_\diamond(\alpha)$ is a pair (X_α, C_α) where $X_\alpha, C_\alpha \subseteq \alpha$, if α is a limit ordinal, then C_α is either a club or the empty set, and $C_\alpha = \varnothing$ when α is not a limit ordinal. We let $F_\diamond(\alpha) = (X_\alpha, C_\alpha)$ be the $<_L$-least pair such that for all $\beta \in C_\alpha$, $X_\beta \neq X_\alpha \cap \beta$ if α is a limit ordinal and such pair exists and otherwise we let $F_\diamond(\alpha) = (\varnothing, \varnothing)$.

- We let $C_\diamond \subseteq \mathbb{ON}$ be the class of all limit ordinals α such that for all $\beta < \alpha$, $F_\diamond \restriction \beta \in L_\alpha$. Notice that for every regular cardinal α, $C_\diamond \cap \alpha$ is a club.

Definition 3.3. For a given regular cardinal α and a subset $A \subseteq \alpha$, we define the sequence $(X_\gamma, C_\gamma)_{\gamma \in A}$ to be $(F_\diamond(\gamma))_{\gamma \in A}$, and the sequence $(X_\gamma)_{\gamma \in A}$ to be the sequence of sets X_γ such that $F_\diamond(\gamma) = (X_\gamma, C_\gamma)$ for some C_γ.

By S^α_μ we denote the set of all μ-cofinal ordinals below α.

Remark. It is known that if α and μ are regular cardinals such that $\mu < \alpha$, then the sequence $(X_\gamma)_{\gamma \in S^\alpha_\mu}$ is a diamond sequence (i.e. for all $Y \subseteq \alpha$, the set $\{\gamma \in S^\alpha_\mu \mid Y \cap \gamma = X_\gamma\}$ is stationary). Notice that if $\beta \in C_\diamond$, then for all $\gamma < \beta$, $X_\gamma \in L_\beta$.

By ZF^- we mean ZFC+$(V = L)$ without the power set axiom. By $\mathrm{ZF}^\diamond$ we mean ZF^- with the following axiom:

> "Let $(S_\gamma, D_\gamma) = F_\diamond(\gamma)$ for all $\gamma < \alpha$ and $\mu < \alpha$ a regular cardinal. Then $(S_\gamma)_{\gamma \in S^\alpha_\mu}$ is a diamond sequence."

Whether or not ZF^- proves $\mathrm{ZF}^\diamond$ is irrelevant for the present argument. We denote by $\mathrm{Sk}(Y)^{L_\theta}$ the Skolem closure of Y in L_θ under the definable Skolem functions.

Lemma 3.4. *($V = L$) For any Σ_1-formula $\varphi(\eta, x)$ with parameter $x \in 2^\kappa$ and a regular cardinal $\mu < \kappa$, then for all $\eta \in 2^\kappa$ we have:*

(i) *If $\varphi(\eta, x)$ holds, then A contains a club,*

(ii) *If $\varphi(\eta, x)$ does not hold, then $S \backslash A$ is μ-stationary,*

where $S = \{\alpha \in S^\kappa_\mu \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$ and

$$A = \big\{\alpha \in C_\diamond \cap \kappa \mid \exists \beta > \alpha \big(L_\beta \models \mathrm{ZF}^\diamond \wedge \varphi(\eta \restriction \alpha, x \restriction \alpha) \wedge r(\alpha)\big)\big\}$$

where $r(\alpha)$ is the formula "α is a regular cardinal".

Remark. This Lemma is reminiscent of [HK, Remark 1.10], but there is a big difference, because now S depends on η through the diamond-sequence which makes this Lemma stronger. The proof in [HK] is not applicable here.

Proof. Let $\mu < \kappa$ be a regular cardinal. Suppose that $\eta \in 2^\kappa$ is such that $\varphi(\eta, x)$ holds. Let $\theta > \kappa$ be a cardinal large enough such that

$$L_\theta \models \mathrm{ZF}^\diamond \wedge \varphi(\eta, x) \wedge r(\kappa).$$

For each $\alpha < \kappa$, let

$$H(\alpha) = \mathrm{Sk}(\alpha \cup \{\kappa, \eta, x\})^{L_\theta}$$

and $\bar{H}(\alpha)$ the Mostowski collapse of $H(\alpha)$. Let

$$D = \{\alpha < \kappa \mid H(\alpha) \cap \kappa = \alpha\}.$$

Then D is a club set and $D \cap C_\diamond$ is a club. Since $H(\alpha)$ is an elementary submodel of L_θ and the Mostowski collapse $\bar{H}(\alpha)$ is equal to L_β for some $\beta > \alpha$, we have $D \cap C_\diamond \subseteq A$. This proves i.

Suppose $\eta \in 2^\kappa$ is such that $\varphi(\eta, x)$ does not hold. Let $\mu < \kappa$ be a regular cardinal. Let C be an arbitrary unbounded set which is closed under μ-limits (a μ-club). We will show that $C \cap (S \setminus A)$ is non-empty which by the arbitrariness of C implies that $S \setminus A$ is μ-stationary, as desired.

Let $\theta > \kappa$ be a large enough cardinal such that

$$L_\theta \models \mathrm{ZF}^\diamond \wedge \neg\varphi(\eta, x) \wedge r(\kappa).$$

Let

$$H(\alpha) = \mathrm{Sk}(\alpha \cup \{\kappa, C, \eta, x, (X_\gamma, C_\gamma)_{\gamma \in S^\kappa_\mu}\})^{L_\theta}.$$

Let

$$D = \{\alpha \in S^\kappa_\mu \mid H(\alpha) \cap \kappa = \alpha\}$$

Then D is an unbounded set, closed under μ-limits. Notice that since $H(\alpha)$ is an elementary substructure of L_θ, then $H(\alpha)$ calculates all cofinalities correctly below α. Let $S = \{\alpha \in S^\kappa_\mu \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$ and α_0 be the least ordinal in $(\lim_\mu D) \cap S$ (where $\lim_\mu D$ is the set of ordinals of D that are μ-cofinal limits of elements of D). By the elementarity of each $H(\alpha)$ we conclude that $\alpha_0 \in C$. It remains to show that $\alpha_0 \notin A$ to complete the proof.

Let $\bar\beta$ be such that $L_{\bar\beta}$ is equal to the Mostowski collapse of $H(\alpha_0)$. Suppose, towards a contradiction, that $\alpha_0 \in A$. Thus $\alpha_0 \in C_\diamond \cap \kappa$ and there exists $\beta > \alpha_0$ such that

$$L_\beta \models \mathrm{ZF}^\diamond \wedge \varphi(\eta \restriction \alpha_0, x \restriction \alpha_0) \wedge r(\alpha_0).$$

Since $\varphi(\eta, x)$ is a Σ_1-formula which holds in L_β and not in $L_{\bar\beta}$, β must be greater than $\bar\beta$. It must be a limit ordinal because $L_\beta \models \mathrm{ZF}^-$.

Claim 3.4.1. *L_β satisfies the following:*

(i) *For all $\gamma \in S \cap \alpha_0$, γ has cofinality μ.*

(ii) *$S \cap \alpha_0$ is a stationary subset of α_0.*

(iii) *$D \cap \alpha_0$ is a μ-club subset of a_0.*

Proof. (i) $H(\alpha_0)$ calculates all cofinalities correctly below α_0. Thus $L_{\bar\beta}$ calculates all cofinalities correctly below α_0. Since β is greater than $\bar\beta$, L_β also calculates all cofinalities correctly below α_0. Since $S \cap \alpha_0 \subseteq S^\kappa_\mu$ in L, we have that $S \cap \alpha_0 \subseteq S^\kappa_\mu$ holds in L_β.

(ii) Since $\alpha_0 \in C_\diamond \cap \kappa$ and L_β satisfies $\mathrm{ZF}^\diamond$ and $r(\alpha_0)$, L_β satisfies that $S \cap \alpha_0$ is a stationary subset of α_0.

(iii) Being unbounded in α_0 is absolute between L and L_β and since $\alpha_0 \in \lim_\mu D$, $D \cap \alpha_0$ is unbounded in α_0, so it remains to show that $D \cap \alpha_0$ is closed under μ-limits in L_β.

Let $\alpha < \alpha_0$ be such that $L_\beta \models \mathrm{cf}(\alpha) = \mu \ \wedge\ \bigcup(D \cap \alpha) = \alpha$, we will show that $L_\beta \models \alpha \in D \cap \alpha_0$. Since L_β calculates all cofinalities correctly below α_0, $L \models \mathrm{cf}(\alpha) = \mu \wedge \bigcup(D \cap \alpha) = \alpha$. D is a μ-club in L, thus $L \models \alpha \in D$. Since $\alpha < \alpha_0$, $L \models \alpha \in D \cap \alpha_0$. We will finish the proof by showing that $L \models \alpha \in D \cap \alpha_0$ implies $L_\beta \models \alpha \in D \cap \alpha_0$.

Notice that $H(\alpha_0)$ is a definable subset of L_θ and D is a definable subset of L_θ. By elementarity, $D\cap\alpha_0$ is a definable subset of $H(\alpha_0)$, we conclude that $D\cap\alpha_0$ is a definable subset of $L_{\bar\beta}$ and $D\cap\alpha_0\in L_\beta$. Therefore $L_\beta\models\alpha\in D\cap\alpha_0$. $\square$

Since $L_\beta\models r(\alpha_0)$, by the previous claim we concluded that L_β satisfies "$\lim_\mu D\cap\alpha_0$ is a μ-club". Since $S\cap\alpha_0$ is a stationary subset of α_0 in L_β, we conclude that

$$L_\beta\models(\lim_\mu D\cap\alpha_0)\cap S\cap\alpha_0\neq\varnothing,$$

so

$$L\models(\lim_\mu D\cap\alpha_0)\cap S\cap\alpha_0\neq\varnothing.$$

This contradicts the minimality of α_0. $\square$

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Suppose Q is a Σ_1^1 quasi-order on κ^κ. Let $a\colon\kappa^\kappa\to 2^{\kappa\times\kappa}$ be the map defined by

$$a(\eta)(\alpha,\beta)=1\Leftrightarrow\eta(\alpha)=\beta.$$

Let b be a continuous bijection from $2^{\kappa\times\kappa}$ to 2^κ, and $c=b\circ a$. Define $Q'\subset 2^\kappa\times 2^\kappa$ by

$$(\eta,\xi)\in Q'\Leftrightarrow(\eta=\xi)\vee(\eta,\xi\in\operatorname{ran}(c)\wedge(c^{-1}(\eta),c^{-1}(\xi))\in Q)$$

So c is a continuous reduction of Q to Q', and Q' is a Σ_1^1 quasi-order because it is a continuous image of Q. On the other hand Q' is a quasi-order on 2^κ and not on κ^κ like the original Q was. Hence, we can assume, without loss of generality, that Q is a quasi-order on 2^κ.

There is a Σ_1-formula of set theory $\psi(\eta,\xi)=\psi(\eta,\xi,x)=\exists k\varphi(k,\eta,\xi,x)\vee\eta=\xi$ with $x\in 2^\kappa$, such that for all $\eta,\xi\in 2^\kappa$,

$$(\eta,\xi)\in Q\Leftrightarrow\psi(\eta,\xi),$$

we added $\eta=\xi$ to $\psi(\eta,\xi)$, to ensure that when we reflect $\psi(\eta\restriction\alpha,\xi\restriction\alpha)$ we get a reflexive relation. Let $r(\alpha)$ be the formula "α is a regular cardinal" and $\psi^Q(\kappa)$ be the sentence with parameter κ that asserts that $\psi(\eta,\xi)$ defines a quasi-order on 2^κ. For all $\eta\in 2^\kappa$ and $\alpha<\kappa$, let

$$T_{\eta,\alpha}=\{p\in 2^\alpha\mid\exists\beta>\alpha(L_\beta\models\mathrm{ZF}^\diamond\wedge\psi(p,\eta\restriction\alpha,x\restriction\alpha)\wedge r(\alpha)\wedge\psi^Q(\alpha))\}.$$

Let $(X_\alpha)_{\alpha\in S^\kappa_\mu}$ be the diamond sequence of Definition 3.3, and for all $\alpha \in S^\kappa_\mu$, let χ_α be the characteristic function of X_α. Define $\mathcal{F}\colon 2^\kappa \to 2^\kappa$ by

$$\mathcal{F}(\eta)(\alpha) = \begin{cases} 1 & \text{if } \chi_\alpha \in T_{\eta,\alpha} \text{ and } \alpha \in S^\kappa_\mu \\ 0 & \text{otherwise} \end{cases}$$

Claim. *If $\eta\ Q\ \xi$, then $T_{\eta,\alpha} \subseteq T_{\xi,\alpha}$ for club-many α's.*

Proof. Suppose $\psi(\eta,\xi,x) = \exists k \varphi(k,\eta,\xi,x)$ holds and let k witness that. Let θ be a cardinal large enough such that $L_\theta \models \mathrm{ZF}^\diamond \wedge \varphi(k,\eta,\xi,x) \wedge r(\kappa)$. For all $\alpha < \kappa$ let $H(\alpha) = \mathrm{Sk}(\alpha \cup \{\kappa,k,\eta,\xi,x\})^{L_\theta}$. The set $D = \{\alpha < \kappa \mid H(\alpha)\cap\kappa = \alpha \wedge H(\alpha) \models \psi^Q(\alpha)\}$ is a club. Using the Mostowski collapse we have that

$$D' = \{\alpha < \kappa \mid \exists\beta > \alpha (L_\beta \models \mathrm{ZF}^\diamond \wedge \varphi(k\restriction\alpha, \eta\restriction\alpha, \xi\restriction\alpha, x\restriction\alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha))\}$$

contains a club. For all $\alpha \in D'$ and $p \in T_{\eta,\alpha}$ we have that

$$\exists\beta_1 > \alpha(L_{\beta_1} \models \mathrm{ZF}^\diamond \wedge \psi(p, \eta\restriction\alpha, x\restriction\alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha))$$

and

$$\exists\beta_2 > \alpha(L_{\beta_2} \models \mathrm{ZF}^\diamond \wedge \psi(\eta\restriction\alpha, \xi\restriction\alpha, x\restriction\alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha)).$$

Therefore, for $\beta = \max\{\beta_1,\beta_2\}$ we have that

$$L_\beta \models \mathrm{ZF}^\diamond \wedge \psi(p,\eta\restriction\alpha, x\restriction\alpha) \wedge \psi(\eta\restriction\alpha,\xi\restriction\alpha,x\restriction\alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha).$$

Since $\psi^Q(\alpha)$ holds and so transitivity holds for $\psi(\eta,\xi)$ in L_β, we conclude that

$$L_\beta \models \mathrm{ZF}^\diamond \wedge \psi(p,\xi\restriction\alpha, x\restriction\alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha)$$

so $p \in T_{\xi,\alpha}$ and $T_{\eta,\alpha} \subseteq T_{\xi,\alpha}$. This holds for all $\alpha \in D'$. □

By the previous claim, we conclude that if $\eta\ Q\ \xi$, then there is a μ-club C such that for every $\alpha \in C$ it holds that $\chi_\alpha \in T_{\eta,\alpha} \Rightarrow \chi_\alpha \in T_{\xi,\alpha}$. Therefore $(\mathcal{F}(\eta)^{-1}\{1\}\backslash\mathcal{F}(\xi)^{-1}\{1\}) \cap C = \varnothing$, and $\mathcal{F}(\eta) \sqsubseteq^\mu \mathcal{F}(\xi)$.

For the other direction, suppose $\neg\psi(\eta,\xi,x)$ holds. Let $S = \{\alpha \in S^\kappa_\mu \mid X_\alpha = \eta^{-1}\{1\}\cap\alpha\}$. Since $(X_\gamma)_{\gamma\in S^\kappa_\mu}$ is a diamond sequence, S is a stationary set. By Lemma 3.4 we know that $S\backslash A$ is stationary, where

$$A = \{\alpha \in C_\diamond \cap \kappa \mid \exists\beta > \alpha(L_\beta \models \mathrm{ZF}^\diamond \wedge \psi(\eta\restriction\alpha, \xi\restriction\alpha, x\restriction\alpha) \wedge r(\alpha))\}.$$

Since for all $\alpha \in S\backslash A$ we have that $X_\alpha = \eta^{-1}\{1\} \cap \alpha$, so $\chi_\alpha \in T_{\eta,\alpha}$. We conclude that for all $\alpha \in S\backslash A$, $\mathcal{F}(\eta)(\alpha) = 1$. On the other hand, for all $\alpha \in S\backslash A$ it holds that

$$\forall \beta > \alpha (L_\beta \not\models \mathrm{ZF}^\diamond \wedge \psi(\eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha))$$

so

$$\forall \beta > \alpha (L_\beta \not\models \mathrm{ZF}^\diamond \wedge \psi(\chi_\alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha)).$$

Therefore

$$\forall \beta > \alpha (L_\beta \not\models \mathrm{ZF}^\diamond \wedge \psi(\chi_\alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha) \wedge \psi^Q(\alpha))$$

we conclude that $\chi_\alpha \notin T_{\xi,\alpha}$, and $\mathcal{F}(\xi)(\alpha) = 0$. Hence, for all $\alpha \in S\backslash A$, $\mathcal{F}(\eta)(\alpha) = 1$ and $\mathcal{F}(\xi)(\alpha) = 0$. Since $S\backslash A$ is stationary, we conclude that $\mathcal{F}(\eta)^{-1}\{1\}\backslash\mathcal{F}(\xi)^{-1}\{1\}$ is stationary and $\mathcal{F}(\eta) \not\sqsubseteq^\mu \mathcal{F}(\xi)$. □

4 Corollaries to Theorem 3.1

4.1 Σ_1^1-completeness of E_μ^2 in L

Theorem 4.1 ($V = L$, $\kappa > \omega$). *$\sqsubseteq^{\mathrm{NS}}$ is a Σ_1^1-complete quasi-order.*

Proof. Follows from Fact 2.4 and Theorem 3.1. □

Theorem 4.2 ($V = L$). *Let μ be a regular cardinal below κ, then E_μ^2 is a Σ_1^1-complete equivalence relation.*

Proof. This follows from Theorem 3.1, because E_μ^2 is a symmetrization of the quasi-order $\sqsubseteq^\mu$. □

The above result cannot be proved in ZFC. It was shown in [FHK, Thm 56] that if κ is not a successor of a singular cardinal, then in a cofinality preserving forcing extension $E_{\mu_1}^2$ and $E_{\mu_2}^2$ are $\leqslant_B$-incomparable for regular cardinals $\mu_1 < \mu_2 < \kappa$.

Theorem 4.1 gives consistently a positive answer to "Given a weakly compact cardinal κ, is $\sqsubseteq^{\mathrm{NS}}$ complete?" [Mot, Q. 11.4]. Theorem 4.2 answers the questions "Is it consistently true that $E_\mu^2 \leqslant_B E_\lambda^2$ for $\lambda < \mu$?" [FHK],[KLLS, Q. 3.47] (take $\lambda = \omega$, $\mu = \omega_1$ and $\kappa = \omega_2$), and gives consistently a positive answer to "Is E_μ^κ Borel-reducible to E_μ^2 for a regular μ?" [FHK15, Q. 15], [KLLS, Q. 3.46].

4.2 Σ^1_1-completeness of $\sqsubseteq_{\mathrm{DLO}}$ and $\sqsubseteq_G$ in L

[Mot, Q. 11.3] asks "Given a weakly compact cardinal κ, is $\sqsubseteq_{\mathrm{DLO}}$ complete for Σ^1_1 quasi-orders? What about arbitrary regular cardinals κ?" In this section we apply Theorem 3.1 to show that the answer is positive if $V = L$. To do that we first have to establish a general theorem about $\sqsubseteq_{\mathrm{DLO}}$:

Theorem 4.3. *Suppose that for all $\lambda < \kappa$ we have $\lambda^\omega < \kappa$. Then there is a continuous reduction of $\sqsubseteq^\omega$ to $\sqsubseteq_{\mathrm{DLO}}$.*

Proof. Fix an ω-club $G \subseteq S^\kappa_\omega \setminus (\omega + 1)$ with the property that for all $\alpha < \kappa$ and all $\beta < \kappa$ there exists $\gamma < \kappa$ with $\beta < \gamma < \kappa$ such that $[\gamma, \gamma + \alpha] \cap G = \varnothing$, where $[\gamma, \gamma + \alpha] = \{\delta < \kappa \mid \gamma \leqslant \delta \leqslant \gamma + \alpha\}$, thus G is in a sense "sparse". Such a G can be obtained by constructing a sequence $(\gamma_\alpha)_{\alpha<\kappa}$ as follows. Let $\gamma_0 = \omega + 1$, for successor $\beta = \alpha + 1$ let $\gamma_\beta = \gamma_\alpha + \gamma_\alpha$ and for limit β let $\gamma_\beta = \sup\{\gamma_\alpha \mid \alpha < \beta\}$. Then let $G = \{\gamma_\alpha \mid \alpha < \kappa\} \cap S^\kappa_\omega$, For a subset $A \subseteq \kappa$, denote

$$A_G = ((A \cap G) \cup (\kappa \setminus G)) \setminus \{\omega\}$$

On the one hand A_G is equivalent to A up to the ω-non-stationary ideal. On the other hand A_G contains arbitrarily long intervals. Note also that A_G contains all ordinals of uncountable cofinality, $S^\kappa_{>\omega} \subset A_G$.

For ordinals α, β, we say that a function $f \colon \alpha \to \beta$ is *continuous* if it is continuous with respect to the order topology, that is, for all increasing sequences $(\gamma_\delta)_{\delta<\lambda} \subset \alpha$, $\lambda < \alpha$, we have

$$\sup\{f(\gamma_\delta) \mid \delta < \lambda\} = f(\sup\{\gamma_\delta \mid \delta < \lambda\}).$$

Claim 4.3.1. *Suppose A and B are subsets of κ. Then $A \setminus B$ is ω-non-stationary if and only if there exists a strictly increasing continuous $f \colon \kappa \to \kappa$ such that*

$$f[A_G] \subseteq B_G$$

Proof. From the definition of A_G and B_G we see that $A_G \setminus B_G = (A \setminus B) \cap G \cap S^\kappa_\omega$. Since G is an ω-club, we have that $A_G \setminus B_G$ is ω-stationary if and only if $A \setminus B$ is. For any $f \colon \kappa \to \kappa$ which is increasing and continuous the set $C_f = \{\alpha < \kappa \mid f(\alpha) = \alpha\}$ is club. Thus, if $A \setminus B$ is ω-stationary, then $(A_G \cap C_f) \setminus B_G = (A_G \setminus B_G) \cap C_f$ is also ω-stationary and therefore non-empty. This proves the direction "from right to left" of the claim.

Assume now that $A \setminus B$ is not ω-stationary and let $C_1 \subseteq S^\kappa_\omega$ be an ω-club such that $A \cap C_1 \subseteq B$. Let $C = C_1 \cap G$. Note that now not only $A \cap C \subseteq B$, but also $A_G \cap C \subseteq B_G$. We will define $f \colon \kappa \to \kappa$ by inductively building a sequence of strictly increasing continuous functions $f_\alpha \in \kappa^{<\kappa}$ such that

(i) if $\alpha < \beta$, then $f_\alpha \subset f_\beta$,

(ii) the domain of f_α is a successor ordinal $\varepsilon + 1$ for some $\varepsilon \in C \cup S^\kappa_{>\omega}$, where $S^\kappa_{>\omega}$ is the set of ordinals below κ with uncountable cofinality.

(iii) if $\alpha < \beta$, then $\operatorname{ran}(f_\alpha) \subset \operatorname{dom}(f_\beta)$,

(iv) if $f_\alpha(\delta) = \delta$, then $\delta \in C \cup S^\kappa_{>\omega}$,

(v) if $f_\alpha(\delta) \neq \delta$, then $f_\alpha(\delta) \in B_G$.

Before constructing this sequence let us show that $f = \bigcup_{\alpha<\kappa} f_\alpha$ will be the desired function. It is strictly increasing and continuous, because f_α are. To show that $f[A_G] \subset B_G$, suppose that $\gamma \in f[A_G]$. Now $\gamma = f(\delta)$ for some $\delta \in A_G$ and clearly $\gamma = f_\alpha(\delta)$ for sufficiently large α. By (5), if $\gamma = f_\alpha(\delta) \neq \delta$, we have $\gamma \in B_G$ and we are done. So we may assume that $\gamma = f_\alpha(\delta) = \delta$. Since $\delta \in A_G$ we now also have $\gamma \in A_G$. By (4) we have $\gamma \in C \cup S^\kappa_{>\omega}$. If $\gamma \in C$, then since $C \cap A_G \subseteq B_G$, we have $\gamma \in B_G$ so we are done. If $\gamma \in S^\kappa_{>\omega}$, then by the definition of G it must be the case that $\gamma \in \kappa \setminus G$. But $\kappa \setminus G \subset B_G$, so again $\gamma \in B_G$.

Thus, it remains to construct the sequence $(f_\alpha)_{\alpha<\kappa}$ satisfying (1)–(5). Let $f_0 = \varnothing$. If f_α is defined, then define $f_{\alpha+1}$ as follows. Let $\varepsilon_\alpha = \max \operatorname{dom} f_\alpha$. By (2) ε_α is well defined and we have $\operatorname{dom} f_\alpha = [0, \varepsilon_\alpha]$ and $\varepsilon_\alpha \in C \cup S^\kappa_{>\omega}$. Let $\varepsilon_{\alpha+1}$ be some ordinal such that $\varepsilon_{\alpha+1} > f_\alpha(\varepsilon_\alpha)$ and $\varepsilon_{\alpha+1} \in C$. Then find $\gamma_0 > \varepsilon_{\alpha+1}$ such that $[\gamma_0, \gamma_0 + \varepsilon_{\alpha+1}] \subset B_G$ which is possible by the definition of G and the fact that $\kappa \setminus G \subset B_G$. Now define for all $\delta \leqslant \varepsilon_{\alpha+1}$

$$f_{\alpha+1}(\delta) = \begin{cases} f_\alpha(\delta) & \text{if } \delta \leqslant \varepsilon_\alpha \\ \gamma_0 + \delta & \text{otherwise,} \end{cases}$$

Clearly (1) and (2) are satisfied for $f_{\alpha+1}$. Since $\gamma_0 > 0$, we have $f_{\alpha+1}(\delta) > \delta$ for all $\delta \in [\varepsilon_\alpha + 1, \varepsilon_{\alpha+1}]$, so if f_α satisfies (4), then so does $f_{\alpha+1}$. Because of the choice of γ_0, (5) is satisfied. Also (3) is satisfied by the choice of $\varepsilon_{\alpha+1}$.

If β is a limit and f_α is defined for $\alpha < \beta$, then let $\varepsilon_\beta = \sup_{\alpha<\beta} \varepsilon_\alpha$. From (3) and that f_α are increasing it follows that

$$\varepsilon_\beta = \sup_{\alpha<\beta} \operatorname{ran} f_\alpha = \sup_{\alpha<\beta} \operatorname{dom} f_\alpha.$$

The domain of f_β is $\varepsilon_\beta + 1$ and it is defined by:

$$f_\beta(\delta) = \begin{cases} f_\alpha(\delta) & \text{if } \delta < \varepsilon_\beta, \text{ for some } \alpha \text{ such that } \delta < \varepsilon_\alpha < \varepsilon_\beta \\ \delta & \text{otherwise,} \end{cases}$$

Clause (3) ensures that this is well-defined. Clauses (1) and (3) are clearly satisfied for f_β. If ε_β has cofinality ω, then (2) is satisfied, because ε_β is the limit of elements of C and C is ω-club. Otherwise, if $\operatorname{cf}(\varepsilon_\beta) > \omega$, then (2) is trivially satisfied. The only new element in $\operatorname{dom} f_\beta$ that is not in $\operatorname{dom} f_\alpha$ for any $\alpha < \beta$ is ε_β and since $\varepsilon_\beta \in C \cup S^\kappa_{>\omega}$, (4) and (5) are satisfied by induction. $\square$

For every $p, q \in \kappa^{\leqslant\omega}$ define $p \prec q$ if either $p \supset q$ or there exists $n < \omega$ such that $p(n) \neq q(n)$ and for the smallest such n we have $p(n) < q(n)$. This defines a linear order on the set $C(\kappa^{\leqslant\omega})$ of all strictly increasing functions $p \in \kappa^{\leqslant\omega}$.

Now for $A \subseteq \kappa$ define the linear order $L(A)$ to be the set

$$\{p \in C(\kappa^{\leqslant\omega}) \mid \operatorname{dom} p = \omega \text{ and } \sup \operatorname{ran} p \in A \text{ and } p(0) = 0\}$$

equipped with the order $\prec$. Note that because elements of $L(A)$ have domain ω, the condition $p \supset q$ in the definition of $p \prec q$ is never relevant for them. This is a modification of a construction given by Baumgartner [Bau]. Clearly $A \subseteq B$ implies $L(A) \subseteq L(B)$. We will show that $A \mapsto L(A_G)$ is a reduction of $\sqsubseteq^\omega$ to $\sqsubseteq_{\mathrm{DLO}}$. By the definition of A_G, the limit ordinals of A_G (which are the only ones that matter in the definition of $L(A_G)$) are all greater ω which ensures that there is no smallest element in $L(A_G)$. Also clearly $L(A_G)$ does not have a greatest element, because $A_G \cap S^\kappa_\omega$ is unbounded and it is dense by the following argument. If $p \prec q$ for elements of $L(A_G)$, then (because $\operatorname{dom} p = \operatorname{dom} q = \omega$) there is $n < \omega$ with $p(n) < q(n)$. Let $p' \colon \omega \to \kappa$ be defined by $p'(k) = p(k)$ for $k \leqslant n$ and $p'(k) = p(k) + 1$ otherwise. Then $p \prec p' \prec q$.

If $f \colon \kappa \to \kappa$ is continuous and strictly increasing with $f(0) = 0$ and $A \subseteq \kappa$ any set, the definition of $L(A)$ implies that

$$\{f \circ p \mid p \in L(A)\} \subseteq L(f[A]).$$

Thus, if $f \colon \kappa \to \kappa$ is continuous and strictly increasing such that $f[A_G] \subseteq B_G$, then $p \mapsto f \circ p$ defines an embedding from $L(A_G)$ into $L(B_G)$. By Claim 4.3.1 such f exists, if $A \sqsubseteq^\omega B$ (we do not lose generality by assuming $f(0) = 0$).

The other direction is essentially a simplification of the proof of Baumgartner Theorem 5.3(ii) [Bau]. If $A \not\sqsubseteq^\omega B$, then, as noted above, also $A_G \not\sqsubseteq^\omega B_G$ and so $A_G \setminus B_G$ is ω-stationary. So it is sufficient to show that for *any* unbounded $A, B \subseteq S^\kappa_\omega$, if $A \setminus B$ is ω-stationary, then $L(A)$ cannot be embedded into $L(B)$.

So suppose that $A \setminus B$ is stationary and assume towards a contradiction that $h \colon L(A) \to L(B)$ preserves the ordering $\prec$. For any $X \subseteq C(\kappa^{\leqslant\omega})$, let $T(X) = \{p \in$

$C(\kappa^{\leqslant\omega}) \mid \exists q \in X(p \subseteq q)\}$. Note that for every strictly increasing $p \in \kappa^{<\omega}$ with $p(0) = 0$, we have $p \in T(L(A))$ and $p \in T(L(B))$. For $g \in T(L(B))$, let

$$\mathrm{Right}(g) = \{f \in L(A) \mid h(f) = g \text{ or } g \prec h(f)\},$$

$$\mathrm{Left}(g) = \{f \in L(A) \mid h(f) \prec g\}.$$

Let

$$\rho(g) = \{f' \in T(\mathrm{Right}(g)) \mid \text{ for all } g' \in T(\mathrm{Right}(g)), \text{ if } g' \prec f', \text{then } f' \subseteq g'\},$$

$$\lambda(g) = \{f' \in T(\mathrm{Left}(g)) \mid \text{ for all } g' \in T(\mathrm{Left}(g)), \text{ if } f' \prec g', \text{then } g' \subseteq f'\}.$$

Note that $\rho(g)$ and $\lambda(g)$ are linearly ordered by $\subset$. Intuitively $\rho(g)$ is a set of "minimal" elements of $T(\mathrm{Right}(g))$ and $\lambda(g)$ is the set of "maximal" elements of $T(\mathrm{Left}(g))$. Now let $C \subseteq S^\kappa_\omega$ be the set of all α satisfying

(i) for all $f \in L(A)$, $\sup\mathrm{ran}(f) < \alpha \iff \sup\mathrm{ran}(h(f)) < \alpha$,

(ii) $A \cap \alpha$ is unbounded in α,

(iii) if $g \in T(L(B))$ and $\sup\mathrm{ran}(g) < \alpha$, then $\sup\{\sup\mathrm{ran}(f) \mid f \in \rho(g)\} < \alpha$ and $\sup\{\sup\mathrm{ran}(f) \mid f \in \lambda(g)\} < \alpha$,

(iv) if $g \in T(L(B))$, $f \in T(\mathrm{Left}(g))$, $\sup\mathrm{ran}(g), \sup\mathrm{ran}(f) < \alpha$, and there exists $\hat{f} \in \mathrm{Left}(g)$ such that $f \prec \hat{f}$ and $\hat{f} \not\subset f$, then there exists such an $\hat{f}$ with $\sup\mathrm{ran}(\hat{f}) < \alpha$,

(v) if $g \in T(L(B))$, $f \in T(\mathrm{Right}(g))$, $\sup\mathrm{ran}(g), \sup\mathrm{ran}(f) < \alpha$, and there exists $\hat{f} \in \mathrm{Right}(g)$ such that $\hat{f} \prec f$ and $f \not\subset \hat{f}$, then there exists such an $\hat{f}$ with $\sup\mathrm{ran}(\hat{f}) < \alpha$,

Our cardinality assumption on κ guarantees that C is a club. We will show that $C \cap A \subseteq B$ which is a contradiction. Let $\alpha \in C \cap A$ and let $f \in L(A)$ be such that $\sup\mathrm{ran}(f) = \alpha$. We will show that $\sup\mathrm{ran}(h(f)) = \alpha$ and so $h(f) \in L(B)$ and $\alpha \in B$. Suppose not. If $\sup\mathrm{ran}(h(f)) < \alpha$, then by (i), $\sup\mathrm{ran}(f) < \alpha$ which is a contradiction. So we can assume that $\sup\mathrm{ran}(h(f)) > \alpha$. Because we assumed that $p(0) = 0$ for all functions in question, there is $n_0 < \omega$ such that $h(f)(n_0) < \alpha \leqslant h(f)(n_0 + 1)$. Let

$$g = h(f) \restriction (n_0 + 1). \tag{$*$}$$

In particular
$$\sup\operatorname{ran}(g) < \alpha. \tag{$**$}$$
For every $m < \omega$, pick $\alpha_m \in A$ such that $f(m) < \alpha_m < \alpha$. Such α_m exists by (ii). Now for each m fix f_m with $\sup\operatorname{ran}(f_m) = \alpha_m$ and $f_m \supset f \restriction (m+1)$. We have two cases: either (A) $\sup\{m < \omega \mid f_m \in \operatorname{Left}(g)\} = \omega$ or (B) $\sup\{m < \omega \mid f_m \in \operatorname{Right}(g)\} = \omega$. We will show that both (A) and (B) lead to a contradiction with (iii).

Claim 4.3.2. *(A) If there are infinitely many $m < \omega$ with $f_m \in \operatorname{Left}(g)$, then for all $m < \omega$ we have $f \restriction (m+1) \in \lambda(g)$ which contradicts (iii).*

Proof. For every m, there is $m' > m$ such that $f_{m'} \in \operatorname{Left}(g)$ and since $f \restriction (m+1) \subseteq f \restriction (m'+1) \subset f_{m'}$, we have that $f \restriction (m+1) \in T(\operatorname{Left}(g))$. Suppose that $f \restriction (m+1) \notin \lambda(g)$ for some m. Then by the definition of $\lambda(g)$, there exists $\hat{f} \in T(\operatorname{Left}(g))$ such that $f \restriction (m+1) \prec \hat{f}$, but $\hat{f} \not\subset f \restriction (m+1)$. We can w.l.o.g. assume that $\hat{f} \in \operatorname{Left}(g)$ and further, by (iv), that $\sup\operatorname{ran}(\hat{f}) < \alpha$.

Since $f \restriction (m+1) \prec \hat{f}$ and $\hat{f} \not\subset f \restriction (m+1)$, there is $n \leqslant m$ such that $\hat{f}(n) > f(n)$ and n is smallest such that $\hat{f}(n) \neq f(n)$. This n witnesses that $f \prec \hat{f}$. So we have $h(f) \prec h(\hat{f})$. The latter implies that for the first $n' < \omega$ with $h(f)(n') \neq h(\hat{f})(n')$ we have $h(\hat{f})(n') > h(f)(n')$. If $n' > n_0$ (n_0 is defined at ($*$)) then $\sup\operatorname{ran}(h(\hat{f})) \geqslant h(\hat{f})(n') > h(f)(n') \geqslant \alpha$, a contradiction with (i). So $n' \leqslant n_0$ and $h(\hat{f})(n') > h(f)(n') = g(n')$, so we have $g \prec h(\hat{f})$. But this implies that $\hat{f} \in \operatorname{Right}(g)$ which is a contradiction again. This proves the claim. □

Claim 4.3.3. *(B) If there are infinitely many $m < \omega$ with $f_m \in \operatorname{Right}(g)$, then for all $m < \omega$ we have $f \restriction (m+1) \in \rho(g)$ which contradicts (iii).*

Proof. For every m, there is $m' > m$ such that $f_{m'} \in \operatorname{Right}(g)$ and since $f \restriction (m+1) \subset f \restriction (m'+1) \subset f_{m'}$, we have that $f \restriction (m+1) \in T(\operatorname{Right}(g))$. Suppose that $f \restriction (m+1) \notin \rho(g)$ for some m. Then by the definition of $\rho(g)$, there exists $\hat{f} \in T(\operatorname{Right}(g))$ such that $\hat{f} \prec f \restriction (m+1)$, but $f \restriction (m+1) \not\subset \hat{f}$. We can again assume that $\hat{f} \in \operatorname{Right}(g)$ and, by (v), that $\sup\operatorname{ran}\hat{f} < \alpha$. There exists $n \leqslant m$ with $\hat{f}(n) < f(n)$ and n is the smallest such that $\hat{f}(n) \neq f(n)$.

The number n witnesses that $\hat{f} \prec f$ and so we must have $h(\hat{f}) \prec h(f)$. The latter implies that for the first $n' < \omega$ with $h(f)(n') \neq h(\hat{f})(n')$ we have $h(\hat{f})(n') < h(f)(n')$. If $n' > n_0$, then $g \subset h(\hat{f})$ and hence $h(\hat{f}) \prec g$ which is a contradiction with $\hat{f} \in \operatorname{Right}(g)$.

So $n' \leqslant n_0$ and $h(\hat{f})(n') < h(f)(n') = g(n')$, and again $h(\hat{f}) \prec g$, contradiction. This proves the claim. □

This completes the proof of Theorem 4.3. $\square$

Theorem 4.4 ($V = L$)**.** *If $\kappa > \omega$ is a regular cardinal which is not the successor of an ω-cofinal cardinal, then $\sqsubseteq_{\mathrm{DLO}}$ is Σ^1_1-complete.*

Proof. By Theorem 3.1 it is sufficient to reduce $\sqsubseteq^\omega$ to $\sqsubseteq_{\mathrm{DLO}}$. But since $V = L$ every cardinal $\kappa > \omega$ which is not the successor of an ω-cofinal cardinal satisfies the assumption of Theorem 4.3. $\square$

Corollary 4.5. *If $\kappa > \omega$ is a regular cardinal which is not the successor of an ω-cofinal cardinal, then the embeddability of graphs $\sqsubseteq_G$ is Σ^1_1-complete.*

Proof. It is a well known folklore that both embeddability and isomorphism of any model class can be coded into graphs (e.g. the authors of [FS] assume this without proof in the countable case). We will sketch a proof for the sake of completeness in the case of linear orders (Theorem 4.6). $\square$

Theorem 4.6. *For every $\kappa \geqslant \omega$ there is a continuous function $F\colon \mathrm{Mod}^\kappa_{\mathrm{DLO}} \to \mathrm{Mod}^\kappa_G$ which preserves both embeddability and isomorphism.*

Proof. Given a linear order $(L, <_L)$ we will construct a graph $F(L, <_L) = (G, R)$ where $R = R(L, <_L)$ is a binary symmetric irreflexive relation on $G = G(L, <_L)$. This construction will be such that it preserves both the embeddability and the isomorphism relations. Moreover it will be easy to see that if F is translated through coding into a function from $\mathrm{Mod}^\kappa_{\mathrm{DLO}}$ to Mod^κ_G it becomes continuous.

The domain of the graph $G = G(L, <_L)$ consists of a copy of the domain of L plus two vertices for every pair $a, b \in L$ such that $a <_L b$. Formally $G = L \cup ({<_L} \times \{0, 1\})$. The relation $R = R(L, <_L)$ is defined so that for every $a <_L b$ the connections between $a, b, ((a, b), 0)$ and $((a, b), 1)$ are as shown in (4.1):

$$\begin{tikzcd}[arrows=dash]
((a,b),0) \arrow[r] \arrow[d] \arrow[dr] & ((a,b),1) \arrow[d] \\
a & b
\end{tikzcd} \tag{4.1}$$

Now any embedding $g\colon L_1 \to L_2$ induces an embedding

$$\hat{g}\colon G(L_1, <_{L_1}) \to G(L_2, <_{L_2})$$

by

$$\hat{g}(a) = \begin{cases} g(a) & \text{if } a \in L_1, \\ ((g(c_1), g(c_2)), 0) & \text{if } a = ((c_1, c_2), 0) \in {<_{L_1}} \times \{0\}, \\ ((g(c_1), g(c_2)), 1) & \text{if } a = ((c_1, c_2), 1) \in {<_{L_1}} \times \{1\}. \end{cases}$$

This $\hat{g}$ is an isomorphism if and only if g is. On the other hand any embedding g from $G(L_1, <_{L_1})$ to $G(L_2, <_{L_2})$ maps elements of L_1 to elements of L_2, because elements of L_k are precisely the elements of $G(L_k, <_{L_k})$ with an infinite $R(L_k, <_{L_k})$-degree, $k \in \{1,2\}$. It is left to the reader to verify that the way the graph is defined ensures that $g \restriction L_1$ is an embedding from $(L_1, <_{L_1})$ to $(L_2, <_{L_2})$. Again this embedding is an isomorphism if and only if g is. □

4.3 Dichotomy for countable first-order theories in L

In [HKM] it was proved that if $V = L$, κ is a successor of an uncountable regular cardinal λ, then $\cong_{T_1} \leqslant_c \cong_{T_2}$ and $\cong_{T_2} \nleqslant_B \cong_{T_1}$ holds for all T_1 classifiable and T_2 non-classifiable. This result can be improved using Theorem 4.2 together with some results from [FHK]:

Theorem 4.7. *([FHK, Thm 86]) Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and T is a stable unsuperstable complete countable theory. Then $E^2_\omega \leqslant_c \cong_T$.* □

Corollary 4.8 ($V = L$)**.** *Suppose that κ is regular and not the successor of an ω-cofinal cardinal and T is a stable unsuperstable complete countable theory. Then $\cong_T$ is a Σ^1_1-complete relation.*

Proof. Follows from Theorems 4.7 and 4.2 and GCH in L. □

Theorem 4.9. *([FHK, Thm 79]) Suppose that $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$.*

(i) *If T is complete unstable or superstable with OTOP, then $E^2_\lambda \leqslant_c \cong_T$.*

(ii) *If $\lambda \geqslant 2^\omega$ and T is complete superstable with DOP, then $E^2_\lambda \leqslant_c \cong_T$.* □

Corollary 4.10 ($V = L$)**.** *Suppose that κ is the successor of a regular uncountable cardinal λ. If T is a non-classifiable complete countable theory, then $\cong_T$ is a Σ^1_1-complete relation.*

Proof. Follows from Theorems 4.2, 4.7, and 4.9. □

By using yet another Theorem from [FHK] we obtain the following dichotomy in L. The class of Δ^1_1 sets consists of sets A such that both A and the complement of A are Σ^1_1 [FHK].

Theorem 4.11 ($V = L$)**.** *Suppose that κ is the successor of a regular uncountable cardinal λ. If T is a countable first-order theory in a countable vocabulary, not necessarily complete, then one of the following holds:*

- $\cong_T$ *is* Δ^1_1.
- $\cong_T$ *is* Σ^1_1*-complete.*

Proof. For this proof it is useful to bare in mind how the isomorphism relation of a theory is defined, Definition 2.3. Sometimes in literature it is defined differently, but these are mutually Borel-bi-reducible (there is a Borel reduction both ways).

It has been shown [FHK, Thm 70] that if a complete theory T is classifiable, then $\cong_T$ is Δ^1_1. So for a complete countable theory T the result follows from Corollary 4.10. Suppose T is not a complete theory. Let $\mathcal{L}$ be the vocabulary of T and $\{T_\alpha\}_{\alpha<2^\omega}$ be the set of all the complete theories in $\mathcal{L}$ that extend T. Notice that $\cong_T = \bigcap_{\alpha<2^\omega} \cong_{T_\alpha}$, therefore if $\cong_{T_\alpha}$ is a Δ^1_1 equivalence relation for all $\alpha<\kappa$, then so is $\cong_T$ since $2^\omega<\kappa$.

Suppose T' is a complete countable theory in $\mathcal{L}$ that extends T such that $\cong_{T'}$ is not a Δ^1_1 equivalence relation. Then T' is a non-classifiable countable theory. By Corollary 4.10 $\cong_{T'}$ is a Σ^1_1-complete equivalence relation. We will show that $\cong_{T'} \leqslant_B \cong_T$ which finishes the proof. Define $\mathcal{F}\colon \kappa^\kappa\to\kappa^\kappa$ by

$$\mathcal{F}(\eta)=\begin{cases}\eta & \text{if } \mathcal{A}_\eta\models T'\\ \xi & \text{otherwise.}\end{cases}$$

where ξ is a fixed element of κ^κ such that $\mathcal{A}_\xi\not\models T'$. Since T' extends T, $\eta \ \cong_{T'}\ \zeta \Leftrightarrow \mathcal{F}(\eta)\ \cong_T\ \mathcal{F}(\zeta)$. To show that $\mathcal{F}$ is Borel, note that

$$\mathcal{F}^{-1}([\eta\restriction\alpha])=\begin{cases}[\eta\restriction\alpha]\backslash\{\zeta\mid \mathcal{A}_\zeta\not\models T'\} & \text{if } \xi\notin[\eta\restriction\alpha]\\ \{\zeta\mid \mathcal{A}_\zeta\not\models T'\}\cup[\eta\restriction\alpha] & \text{if } \xi\in[\eta\restriction\alpha].\end{cases}$$

Since $[\eta\restriction\alpha]$ is a basic open set and $\{\zeta\mid\mathcal{A}_\zeta\not\models T'\}$ is a Borel set, $[\eta\restriction\alpha]\backslash\{\zeta\mid\mathcal{A}_\zeta\not\models T'\}$ and $[\eta\restriction\alpha]\cup\{\zeta\mid\mathcal{A}_\zeta\not\models T'\}$ are Borel sets. $\square$

The dichotomy of Theorem 4.11 is not provable in ZFC. In [HSa, HSb] it was shown, assuming κ is a successor and $\kappa\in I[\kappa]$, that there is a stable unsuperstable countable theory T in a countable vocabulary such that $\cong_T$ is Borel* (a generalization of Borel sets to non-well-founded trees [FHK, HaSh]). By Theorem 4.7 $\cong_T$ is not Δ^1_1, if E^2_ω is not. It was proved in [HK18] that there is a model of ZFC where Borel$^*\subsetneq\Sigma^1_1$ (unlike in L, [HK]), in which E^2_ω is not Δ^1_1, and in which $\kappa\in I[\kappa]$ for successor κ.

5 The case $V \neq L$

5.1 Σ^1_1-completeness of $\sqsubseteq^{\mathrm{NS}}$ for weakly ineffable κ

In Section 4 we answered the questions [KLLS, Q. 3.47], [Mot, Q.'s 11.3 and 11.4] and [FHK15, Q. 15] under $V = L$. We used Theorem 4.2 as the starting point. But what if $V \neq L$? In this section we provide further partial answers to [Mot, Q.'s 11.3 and 11.4] outside of L. Recall that these questions ask "Given a weakly compact cardinal κ, are $\sqsubseteq^{\mathrm{NS}}$ and $\sqsubseteq_{\mathrm{DLO}}$ complete for Σ^1_1 quasi-orders?" Recall that $\sqsubseteq_G$ is the embeddability of graphs, Definition 2.3. We will use the following theorem:

Theorem 5.1. *([Mot, Cor 10.24]) If κ is weakly compact, then both the quasi-order of embeddability and the equivalence relation of bi-embeddability of graphs, $\sqsubseteq_G$ and $\approx_G$ respectively, are Σ^1_1-complete.*

Definition 5.2 (Weakly compact diamond)**.** Let $\kappa > \omega$ be a cardinal. The *weakly compact ideal* is generated by the sets of the form $\{\alpha < \kappa \mid \langle V_\alpha, \in, U \cap V_\alpha\rangle \models \neg\varphi\}$ where $U \subseteq V_\kappa$ and φ is a Π^1_1-sentence such that $\langle V_\kappa, \in, U\rangle \models \varphi$. A set $A \subseteq \kappa$ is said to be *weakly compact*, if it does not belong to the weakly compact ideal. Note that κ is weakly compact if and only if there exists $A \subseteq \kappa$ which is weakly compact, i.e. the weakly compact ideal is proper. For weakly compact $S \subseteq \kappa$, the *S-weakly compact diamond*, $\mathrm{WC}_\kappa(S)$, is the statement that there exists a sequence $(A_\alpha)_{\alpha<\kappa}$ such that for every $A \subseteq S$ the set

$$\{\alpha < \kappa \mid A \cap \alpha = A_\alpha\}$$

is weakly compact. We denote $\mathrm{WC}_\kappa = \mathrm{WC}_\kappa(\kappa)$.

Weakly compact diamond was originally introduced in [Sun] and thoroughly analyzed in [Hell]. In [AHKM] it was used to study the reducibility properties of E^κ_{reg}. It has been sometimes called the *dual diamond*.

Fact 5.3. If κ is weakly ineffable (same as almost ineffable), then WC_κ holds. See [Hell] for proofs and references.

The proof of Lemma 5.4 can be found in [AHKM] in complete detail.

Lemma 5.4. *Let κ be a weakly compact cardinal. The weakly compact diamond WC_κ implies the following principle WC^*_κ. There exists a sequence $\langle f_\alpha\rangle_{\alpha\in\mathrm{reg}(\kappa)}$ such that*

- $f_\alpha \colon \alpha \to \alpha$,
- *for all $g \in \kappa^\kappa$ and stationary $Z \subseteq \kappa$ the set*

$$\{\alpha \in \mathrm{reg}(\kappa) \mid g \restriction \alpha = f_\alpha \wedge \alpha \cap Z \textit{ is stationary}\}$$

is stationary. □

Let us introduce a version of WC^*_κ for graphs, denoted WC^*_G. Let $G_{<\kappa} = \{\mathcal{G}_\beta\}_{\beta<\kappa}$ be an enumeration of all graphs with domain some ordinal $\alpha < \kappa$. For all $\alpha < \kappa$, let $G_{<\alpha} = \{\mathcal{G}_\beta\}_{\beta<\alpha}$.

The principle WC^*_G states that there exists a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ such that

- $f_\alpha \in (G_{<\alpha})^\alpha$,
- if (S, g) is a pair such that $S \subseteq \kappa$ is stationary and $g \in (G_{<\kappa})^\kappa$, the set

$$\{\alpha \in \mathrm{reg}(\kappa) \mid g \restriction \alpha = f_\alpha \wedge S \cap \alpha \text{ is stationary}\}$$

is stationary.

Fact 5.5. If WC^*_κ holds, then WC^*_G holds.

Proof. Let $\langle \bar{f}_\alpha \rangle_{\alpha<\kappa}$ be a sequence that witnesses WC^*_κ. Define the sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ by $f_\alpha(\beta) = \mathcal{G}_{\bar{f}_\alpha(\beta)}$.

To show that $\langle f_\alpha \rangle_{\alpha<\kappa}$ witnesses WC^*_G, let $g \in (G_{<\kappa})^\kappa$ be any function and $S \subseteq \kappa$ a stationary set. There is a function $\bar{g} \colon \kappa \to \kappa$ such that $g(\alpha) = \mathcal{G}_{\bar{g}(\alpha)}$. Because of WC^*_κ we know that the set

$$\{\alpha \in \mathrm{reg}(\kappa) \mid \bar{g} \restriction \alpha = \bar{f}_\alpha \wedge Z \cap \alpha \text{ is stationary}\}$$

is stationary. From the definitions of $\langle f_\alpha \rangle_{\alpha<\kappa}$ and $\bar{g}$ it follows that the set

$$\{\alpha \in \mathrm{reg}(\kappa) \mid g \restriction \alpha = f_\alpha \wedge Z \cap \alpha \text{ is stationary}\}$$

is stationary. □

Definition 5.6. A *closure point* of a function $s \colon \kappa \to \kappa$ is an ordinal $\alpha < \kappa$ such that for all $\beta < \alpha$ we have $s(\beta) < \alpha$. The set of all closure points of s is a club.

Theorem 5.7. *If κ is weakly compact and WC^*_G holds, then $\sqsubseteq^{\mathrm{reg}}$ as well as $\sqsubseteq^{\mathrm{NS}}$ are Σ^1_1-complete.*

Proof. The claim for $\sqsubseteq^{\mathrm{NS}}$ follows from Fact 2.4 once we prove the claim for $\sqsubseteq^{\mathrm{reg}}$. By Theorem 5.1 it is enough to show that $\sqsubseteq_G \leqslant_B \sqsubseteq^{\mathrm{reg}}$. For all $K, H \in G_{<\kappa}$ we write $K \sqsubseteq H$ if K is embeddable to H. Let us denote by Q the quasi-order $((G_{<\kappa})^\kappa, \leqslant_Q)$, where $f \leqslant_Q g$ holds if there is a club C such that for all $\alpha \in C$, $f(\alpha) \sqsubseteq g(\alpha)$ holds.

Note that every $\mathcal{G}_\alpha$ equals some $\mathcal{A}_p$, $p \in \kappa^{<\kappa}$, where $\mathcal{A}_p$ is as defined in Definition 2.2. Vice versa, if $\eta \in \mathrm{Mod}^\kappa_G$ (i.e. is a code for a graph, that is, $\mathcal{A}_\eta$ is a graph, Definition 2.1), then for every $\alpha < \kappa$ there is $\beta < \kappa$ such that $\mathcal{A}_{\eta\restriction\alpha} = \mathcal{G}_\beta$. Let H be the graph with domain 2 and no edges. Define $F\colon \mathrm{Mod}^\kappa_G \to (G_{<\kappa})^\kappa$ by

$$F(\eta)(\alpha) = \begin{cases} \mathcal{A}_{\eta\restriction\alpha} & \text{if } \alpha \in C_\pi \\ H & \text{otherwise.} \end{cases}$$

where C_π is as in Definition 2.2.

Claim 5.7.1. *$\eta \sqsubseteq_G \xi$ if and only if $F(\eta) \leqslant_Q F(\xi)$.*

Proof. Let us show that if $\eta \sqsubseteq_G \xi$, then $F(\eta) \leqslant_Q F(\xi)$. Suppose $\eta \sqsubseteq_G \xi$, then there is an embedding $f\colon \kappa \to \kappa$ of $\mathcal{A}_\eta$ to $\mathcal{A}_\xi$. Let D be the set of all closure points of f (Definition 5.6). Since D is a club, $f \restriction \alpha$ is an embedding of $\mathcal{A}_{\eta\restriction\alpha}$ to $\mathcal{A}_{\xi\restriction\alpha}$, for all $\alpha \in D \cap C_\pi$. We conclude that $F(\eta) \leqslant_Q F(\xi)$. Let us show that if $\eta \not\sqsubseteq_G \xi$, then $F(\eta) \not\leqslant_Q F(\xi)$. Suppose $\eta \not\sqsubseteq_G \xi$. The property

"(there is no embedding of $\mathcal{A}_\eta$ to $\mathcal{A}_\xi$) $\wedge$ (κ is regular) $\wedge$ (C_π is unbounded)"

is a Π^1_1-property of the structure $(V_\kappa, \in, A)$, where $A = (\eta \times \{0\}) \cup (\xi \times \{1\}) \cup (C_\pi \times \{2\})$. Since κ is weakly compact, there is stationary many ordinals γ such that the above property holds with η and ξ replaced by $\eta \restriction \gamma$ and $\xi \restriction \gamma$ as well as κ replaced by γ, i.e.

"(there is no embedding of $\mathcal{A}_{\eta\restriction\gamma}$ to $\mathcal{A}_{\xi\restriction\gamma}$) $\wedge$ (γ is regular) $\wedge$ ($C_\pi \cap \gamma$ is unbounded in γ)".

We conclude that there are stationary many ordinals γ such that $F(\eta)(\gamma) \not\sqsubseteq F(\xi)(\gamma)$, hence $F(\eta) \not\leqslant_Q F(\xi)$. □

Let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence that witnesses WC^*_G. For all $\alpha \in \mathrm{reg}(\kappa)$ define the relation $\leqslant^\alpha_Q$ on $(G_{<\kappa})^\alpha$ by: $f \leqslant^\alpha_Q g$ if there is a club $C \subseteq \alpha$ such that for all $\beta \in C$, $f(\beta) \sqsubseteq g(\beta)$ holds. Notice that since the intersection of two clubs is a club, then $\leqslant^\alpha_Q$ is a quasi-order. Define the map $\mathcal{F}\colon (G_{<\kappa})^\kappa \to 2^\kappa$ by

$$\mathcal{F}(f)(\alpha) = \begin{cases} 0 & \text{if } f \restriction \alpha \leqslant^\alpha_Q f_\alpha \\ 1 & \text{otherwise} \end{cases}$$

Claim 5.7.2. *$f \leqslant_Q g$ if and only if $\mathcal{F}(f) \sqsubseteq^{\mathrm{reg}} \mathcal{F}(g)$.*

Proof. Let us show that if $f \leqslant_Q g$, then $\mathcal{F}(f) \sqsubseteq^{\mathrm{reg}} \mathcal{F}(g)$. Suppose $f \leqslant_Q g$, then there is a club $C \subseteq \kappa$ such that for all $\alpha \in C$, $f(\alpha) \sqsubseteq g(\alpha)$. Therefore, for all $\alpha \in \lim(C) \cap \mathrm{reg}(\kappa)$ it holds that $f \restriction \alpha \leqslant_Q^\alpha g \restriction \alpha$. Now if $\alpha \in \lim(C) \cap \mathrm{reg}(\kappa)$ is such that $\mathcal{F}(g)(\alpha) = 0$, then $g \restriction \alpha \leqslant_Q^\alpha f_\alpha$, so $f \restriction \alpha \leqslant_Q^\alpha f_\alpha$ and $\mathcal{F}(f)(\alpha) = 0$. We conclude that $(\mathcal{F}(f)^{-1}[1] \backslash \mathcal{F}(g)^{-1}[1]) \cap \mathrm{reg}(\kappa)$ is non-stationary. Hence $\mathcal{F}(f) \sqsubseteq^{\mathrm{reg}} \mathcal{F}(g)$. Let us show that if $f \not\leqslant_Q g$, then $\mathcal{F}(f) \not\sqsubseteq^{\mathrm{reg}} \mathcal{F}(g)$. Suppose that $f \not\leqslant_Q g$, then there is a stationary set $S \subseteq \kappa$ such that for all $\alpha \in S$, $f(\alpha) \not\sqsubseteq g(\alpha)$. Because of WC^*_G we know that the set

$$A = \{\alpha \in \mathrm{reg}(\kappa) \mid g \restriction \alpha = f_\alpha \wedge S \cap \alpha \text{ is stationary}\}$$

is a stationary set. Therefore, for all $\alpha \in A$, $\mathcal{F}(g)(\alpha) = 0$, and for all $\beta \in S \cap \alpha$, $f(\beta) \not\sqsubseteq g(\beta)$. Since for all $\alpha \in A$, $g \restriction \alpha = f_\alpha$, and $S \cap \alpha$ is stationary, we conclude that $f \restriction \alpha \not\leqslant_Q^\alpha f_\alpha$ holds for all $\alpha \in A$. Hence, for all $\alpha \in A$, $\mathcal{F}(g)(\alpha) = 0$ and $\mathcal{F}(f)(\alpha) = 1$. We conclude that $A \subseteq (\mathcal{F}(f)^{-1}[1] \backslash \mathcal{F}(g)^{-1}[1]) \cap \mathrm{reg}(\kappa)$, and since A is stationary, $\mathcal{F}(f) \not\sqsubseteq^{\mathrm{reg}} \mathcal{F}(g)$. □

Clearly $\mathcal{F} \circ F\colon \mathrm{Mod}^\kappa_G \to 2^\kappa$ is a Borel-reduction of $\sqsubseteq_G$ to $\sqsubseteq^{\mathrm{reg}}$. □

Theorem 5.8. *If κ is weakly ineffable, then $\sqsubseteq^{\mathrm{NS}}$ is Σ^1_1-complete.*

Proof. Follows from Fact 5.3, Lemma 5.4, Fact 5.5, and Theorem 5.7. □

Thus, the only case concerning [Mot, Q. 11.4] that is still open is the case where $V \neq L$ and κ is a weakly compact, but not weakly ineffable cardinal. For example the first weakly compact is such [Fri, Lemma 1.12]. For successor cardinals, we know from [FWZ] that it can be forced the relation E^2_{NS} to be a Δ^1_1 equivalence relation. So it is consistently true that $\sqsubseteq^{\mathrm{NS}}$ is not Σ^1_1-complete.

5.2 Σ^1_1-completeness of $\cong_{\mathrm{DLO}}$ and $\cong_G$ for weakly compact κ

In this section we prove:

Theorem 5.9. *Suppose that κ is weakly compact. Then the isomorphism relation on dense linear orders is Σ^1_1-complete.*

Before proving Theorem 5.9, we first prove the following:

Lemma 5.10. *If κ is weakly compact, then the bi-embeddability of graphs $\approx_G$ is reducible to E^κ_{reg} (Definition 2.3).*

Proof. Let C_π be the club as in Definition 2.2 and for all $\alpha \in C_\pi$ define the relation $\approx^\alpha_G$ as follows. For all $\eta, \xi \in \text{Mod}^\kappa_G$, let $\eta \ \approx^\alpha_G \ \xi$, if $\mathcal{A}_{\eta\restriction\alpha}$ is embeddable in $\mathcal{A}_{\xi\restriction\alpha}$ and $\mathcal{A}_{\eta\restriction\alpha}$ is embeddable in $\mathcal{A}_{\xi\restriction\alpha}$ (Definition 2.2).

There are at most κ many equivalence classes of $\approx^\alpha_G$, so let $g_\alpha\colon \text{Mod}^\kappa_G \to \kappa$ be a function with the property that for all $\eta, \xi \in \text{Mod}^\kappa_G$ we have $g_\alpha(\eta) = g_\alpha(\xi)$ if and only if $\eta \approx^\alpha_G \xi$.

Define the reduction $\mathcal{F}\colon \text{Mod}^\kappa_G \to \kappa^\kappa$ by

$$\mathcal{F}(\eta)(\alpha) = \begin{cases} g_\alpha(\eta) & \text{if } \alpha \in C_\pi \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that if $\eta \approx_G \xi$, then $(\mathcal{F}(\eta), \mathcal{F}(\xi)) \in E^\kappa_{\text{reg}}$. Suppose that $\eta \approx_G \xi$. Then there are embeddings $F_1\colon \kappa \to \kappa$ and $F_2\colon \kappa \to \kappa$ from $\mathcal{A}_\eta$ to $\mathcal{A}_\xi$, and from $\mathcal{A}_\xi$ to $\mathcal{A}_\eta$ respectively. Let D_1 and D_2 be the sets of all closure points (Definition 5.6) of F_1 and F_2 respectively. These are closed unbounded sets in κ. Then for all $\alpha \in D_1 \cap D_2 \cap C_\pi$, $\mathcal{A}_{\eta\restriction\alpha}$ and $\mathcal{A}_{\xi\restriction\alpha}$ are bi-embeddable. Hence for all $\alpha \in D_1 \cap D_2 \cap C_\pi$, $\mathcal{F}(\eta)(\alpha) = \mathcal{F}(\xi)(\alpha)$. We conclude that $(\mathcal{F}(\eta), \mathcal{F}(\xi)) \in E^\kappa_{\text{reg}}$.

Let us show that if $\eta \not\approx_G \xi$, then $\mathcal{F}(\eta)$ and $\mathcal{F}(\xi)$ are not E^κ_{reg}-equivalent. Suppose that $(\eta, \xi) \notin \approx_G$, without loss of generality, suppose that there is no embedding of $\mathcal{A}_\eta$ into $\mathcal{A}_\xi$. The property

There is no embedding of $\mathcal{A}_\eta$ to $\mathcal{A}_\xi$ $\wedge$ κ is regular $\wedge$ C_π is unbounded

is a Π^1_1-property of the structure $(V_\kappa, \in, A)$, where $A = (\eta \times \{0\}) \cup (\xi \times \{1\}) \cup (C_\pi \times \{2\})$. Since κ is weakly compact, there are stationary many ordinals $\gamma < \kappa$ such that $C_\pi \cap \gamma$ is unbounded, $\gamma \in C_\pi$, γ is regular, and *there is no embedding of $\mathcal{A}_{\eta\restriction\gamma}$ to $\mathcal{A}_{\xi\restriction\gamma}$*. We conclude that there are stationary many regular cardinals γ with $\mathcal{F}(\eta)(\gamma) \neq \mathcal{F}(\xi)(\gamma)$, hence η and ξ are not E^κ_{reg}-equivalent. □

Corollary 5.11. *If κ is weakly compact, then E^κ_{reg} is Σ^1_1-complete.*

Proof. Follows from Theorem 5.1 and Lemma 5.10. □

Now we can prove Theorem 5.9:

Proof of Theorem 5.9. By [AHKM, Thm 3.9] we have $E^\kappa_{\text{reg}} \ \leqslant_c \cong_{\text{DLO}}$, so the result follows from Corollary 5.11 □

By Theorem 4.6 we get the following corollary to Theorem 5.9:

Corollary 5.12. *Suppose that κ is weakly compact. Then the isomorphism relation on graphs is Σ_1^1-complete.* □

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