Reflection principles and the generalized Baire spaces

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Outline

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The spectrum problem

Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T,\alpha)$?

- **Löwenheim-Skolem Theorem:** $\exists\alpha \geq \omega \ I(T,\alpha) \neq 0 \Rightarrow \forall\beta \geq \omega \ I(T,\beta) \neq 0$.
- **Morley's categoricity:** $\exists\alpha > \omega \ I(T,\alpha) = 1 \Rightarrow \forall\beta > \omega \ I(T,\beta) = 1$
- **Shelah's Main Gap Theorem:** Either, for every uncountable cardinal α, $I(T,\alpha) = 2^\alpha$, or $\forall\alpha > 0 \ I(T,\aleph_\alpha) < \beth_{\omega_1}(\mid\alpha\mid)$.

Approaches

- Shelah's stability theory.
  Classify the models of T by cardinal invariants and clearly differentiate between the theories that can be classified and those that cannot.

- Descriptive set theory.
  It uses Borel-reducibility and the isomorphism relation to define a partial order on the set of all first-order complete countable theories.

The topology

κ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

κ-Borel

The collection of κ-Borel subsets of κ^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ.

A function $f\colon \kappa^\kappa \to \kappa^\kappa$ is *Borel*, if for every open set $A \subseteq \kappa^\kappa$ the inverse image $f^{-1}[A]$ is a Borel subset of κ^κ.

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

We write $E_1 \leqslant^\kappa_B E_2$.

Coding structures

Fix a language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in \kappa^\kappa$ define the structure $\mathcal{A}_f$ with domain κ and for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

Definition (The isomorphism relation)

Given T a first-order complete countable theory in a countable vocabulary, we say that $f, g \in \kappa^\kappa$ are $\cong_T^\kappa$ equivalent if

- $\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
  or
- $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

The complexity

We can define a partial order on the set of all first-order complete countable theories

$$T \leqslant_\kappa^\kappa T' \ iff \ \cong_T^\kappa \leqslant_B^\kappa \cong_{T'}^\kappa$$

The subspace 2^κ

In the subspace 2^κ, we can define the following notions in the same way:

- The bounded topology (the relative subspace topology).
- $E_1 \leqslant^2_B E_2$.
- $f \cong^2_T g$.
- $T \leqslant^2_\kappa T'$.

Reductions

For $X, Y \in \{\kappa^\kappa, 2^\kappa\}$, we say that a function $f\colon X \to Y$ is *Borel*, if for every open set $A \subseteq Y$ the inverse image $f^{-1}[A]$ is a Borel subset of X.

Let E_1 and E_2 be equivalence relations on X and Y respectively. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f\colon X \to Y$ that satisfies $(\eta, \xi) \in E_1 \Leftrightarrow (f(\eta), f(\xi)) \in E_2$. It is denoted by $E_1 \leqslant_B E_2$.

Outline

Shelah's Main Gap Theorem

Theorem (Shelah)

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Question:
Is there a Borel reducibility counterpart of the Main Gap Theorem in the spaces κ^κ and 2^κ?

A Borel reducibility counterpart of the Main Gap

Theorem (Hyttinen, Kulikov, M.)

Suppose that $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$ *and* $\lambda^{<\lambda} = \lambda$. *Then the following statements are consistent:*
If T_1 *is classifiable and* T_2 *is not, then there is an embedding of* $(\mathcal{P}(\kappa), \subseteq)$ *to* $(B^*(T_1, T_2), \leqslant_B)$, *where* $B^*(T_1, T_2)$ *is the set of all Borel*-equivalence relations strictly between* $\cong^2_{T_1}$ *and* $\cong^2_{T_2}$.

Theorem (Hyttinen, Kulikov, M.)

Suppose $\kappa = \lambda^+$ and $\lambda^\omega = \lambda$. If T is a classifiable theory and T' is a stable unsuperstable theory, then $\cong^2_T \leqslant_B \cong^2_{T'}$.

Theorem (M.)

Suppose T is a classifiable theory, T' is a superstable theory with the S-DOP, $\lambda \geq 2^\omega$, and κ an inaccessible cardinal. Then $\cong^\kappa_T \leqslant_B \cong^\kappa_{T'}$

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$E^{\kappa}_{\lambda\text{-club}}$ and $E^{2}_{\lambda\text{-club}}$

For every regular cardinal $\lambda < \kappa$, the relations $E^{\kappa}_{\lambda\text{-club}}$ and $E^{2}_{\lambda\text{-club}}$ are defined as follow.

Definition

- *On the space κ^κ, we say that $f, g \in \kappa^\kappa$ are $E^{\kappa}_{\lambda\text{-club}}$ equivalent if the set $\{\alpha < \kappa | f(\alpha) = g(\alpha)\}$ contains an unbounded set closed under λ-limits.*
- *On the space 2^κ, we say that $f, g \in 2^\kappa$ are $E^{2}_{\lambda\text{-club}}$ equivalent if the set $\{\alpha < \kappa | f(\alpha) = g(\alpha)\}$ contains an unbounded set closed under λ-limits.*

$\diamond$-reflection

Definition

Let X, Y be subsets of κ and suppose Y consists of ordinals of uncountable cofinality. We say that X $\diamond$-reflects to Y if there exists a sequence $\langle D_\alpha \rangle_{\alpha \in Y}$ such that:

- *$D_\alpha \subset \alpha$ is stationary in α.*
- *if $Z \subset X$ is stationary, then $\{\alpha \in Y | D_\alpha = Z \cap \alpha\}$ is stationary.*

Theorem (Friedman, Hyttinen, Kulikov)

$E^2_{\lambda\text{-club}} \leqslant_B E^2_{\lambda^+\text{-club}}$ is consistently true.

Full reflection

Definition

For stationary subsets S and A of κ, we say that S reflects fully in A if the set $\{\alpha \in A \mid S \cap \alpha \text{ is nonstationary in } \alpha\}$ is nonstationary.

Proposition

If every stationary set $S \subset S^\kappa_\gamma$ reflects fully in S^κ_λ, then $E^\kappa_{\gamma\text{-club}} \leqslant_B E^\kappa_{\lambda\text{-club}}$.

Proof

Definition

For every $\alpha < \kappa$ *with* $\gamma < cf(\alpha)$ *define* $E^{\kappa}_{\gamma\text{-}club} \restriction \alpha$ *by:*

$$E^{\kappa}_{\gamma\text{-}club} \restriction \alpha = \{(\eta, \xi) \in \kappa^{\kappa} \times \kappa^{\kappa} \mid \exists C \subseteq \alpha \text{ a } \gamma\text{–}club, \forall \beta \in C, \eta(\beta) = \xi(\beta)\}.$$

$$F(\eta)(\alpha) = \begin{cases} f_{\alpha}(\eta), \text{ if } cf(\alpha) = \lambda \\ 0, \text{ otherwise.} \end{cases}$$

where $f_{\eta}(\alpha)$ is a code in $\kappa \backslash \{0\}$ for the $(E^{\kappa}_{\gamma\text{-club}} \restriction \alpha)$–equivalence class of η.

Full reflection

Theorem (Jech, Shelah)

Let $\kappa_2 < \kappa_3 < \cdots < \kappa_n < \cdots$ be a sequence of supercompact cardinals. There is a generic extension $V[G]$ in which $\kappa_n = \aleph_n$ for all $n \geq 2$ and such that:

1. *Every stationary set $S \subset S^{\omega_2}_{\omega}$ reflects fully in $S^{\omega_2}_{\omega_1}$.*
2. *For every $2 < n$ and every $0 \leqslant k \leqslant n-3$, every stationary set $S \subset S^{\omega_n}_{\omega_k}$ reflects fully in $S^{\omega_n}_{\omega_{n-1}}$.*

Corollary

Let $\kappa_2 < \kappa_3 < \cdots < \kappa_n < \cdots$ be a sequence of supercompact cardinals. There is a generic extension $V[G]$ in which

1. $E^{\omega_2}_{\omega\text{-club}} \leqslant_B E^{\omega_2}_{\omega_1\text{-club}}$.
2. *For every $2 < n$ and every $0 \leqslant k \leqslant n-3$, $E^{\omega_n}_{\omega_k\text{-club}} \leqslant_B E^{\omega_n}_{\omega_{n-1}\text{-club}}$.*

Indescribable Cardinals

Theorem

Suppose κ is a $\Pi_1^{\lambda^+}$-indescribable cardinal and that $V = L$. Then there is a forcing extension where κ is collapsed to λ^{++} and $E^{\lambda^{++}}_{\lambda\text{-club}} \leqslant_B E^2_{\lambda^+\text{-club}}$.

Σ_1^1-completeness

Definition

An equivalence relation E on $X \in \{\kappa^\kappa, 2^\kappa\}$ is Σ_1^1 if E is the projection of a closed set in $\kappa^\kappa \times X$ and it is Σ_1^1-complete, if every Σ_1^1 equivalence relation is Borel reducible to it.

Theorem (Hyttinen, Kulikov)

Suppose $V = L$ and $\kappa > \omega$. Then $E^\kappa_{\mu\text{-club}}$ is Σ_1^1-complete for every regular $\mu < \kappa$.

Σ_1^1-completeness

Definition

For κ a Mahlo cardinal, the relation E^{κ}_{reg} is defined in the space $\kappa^{\kappa} \times \kappa^{\kappa}$ by:

$$(\eta, \xi) \in E^{\kappa}_{reg} \Leftrightarrow \{\alpha \in reg(\kappa) \mid \eta(\alpha) \neq \xi(\alpha)\} \text{ is not stationary }.$$

Definition

For κ a Mahlo cardinal, the relation E^{2}_{reg} is defined in the space $2^{\kappa} \times 2^{\kappa}$ by:

$$(\eta, \xi) \in E^{2}_{reg} \Leftrightarrow \{\alpha \in reg(\kappa) \mid \eta(\alpha) \neq \xi(\alpha)\} \text{ is not stationary }.$$

Σ^1_1-completeness

Theorem

If κ is a Π^1_2-indescribable cardinal, then E^κ_{reg} is Σ^1_1-complete.

Theorem

Suppose κ is a supercompact cardinal. There is a generic extension $V[G]$ in which $E^\kappa_{reg} \leqslant_B E^2_{reg}$ holds and κ is still supercompact in the extension.

Corollary

Suppose κ is a supercompact cardinal. There is a generic extension $V[G]$ in which E^2_{reg} is Σ^1_1-complete.

Corollary

Let DLO be the theory of dense linear orderings without end points. If κ is a Π^1_2–indescribable cardinal, then $\cong_{DLO}$ is Σ^1_1-complete.

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