The Equivalence Modulo Non-stationary Ideals and Shelah's Main Gap Theorem

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Outline

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1 Classifying First-order countable Theories

2 The Equivalence Modulo Non-stationary Ideals

The spectrum problem

Let $I(T, \alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T, \alpha)$?

- **Löwenheim-Skolem Theorem:** $\exists \alpha \geq \omega \; I(T, \alpha) \neq 0 \Rightarrow \forall \beta \geq \omega \; I(T, \beta) \neq 0$.
- **Morley's categoricity:** $\exists \alpha > \omega \; I(T, \alpha) = 1 \Rightarrow \forall \beta > \omega \; I(T, \beta) = 1$
- **Shelah's Main Gap Theorem:** Either, for every uncountable cardinal α, $I(T, \alpha) = 2^{\alpha}$, or $\forall \alpha > 0 \; I(T, \aleph_\alpha) < \beth_{\omega_1}(| \alpha |)$.

Approaches

- Shelah's stability theory.
  Classify the models of T by cardinal invariants and clearly differentiate between the theories that can be classified and those that cannot.

- Descriptive set theory:
  It uses Borel-reducibility and the isomorphism relation to define a partial order on the set of all first-order complete countable theories.

The topology

κ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *continuous reducible* to E_2, if there is a continuous function $f: \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

We write $E_1 \leqslant_c^\kappa E_2$.

Coding structures

Fix a language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in \kappa^\kappa$ define the structure $\mathcal{A}_f$ with domain κ by: for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

Definition (The isomorphism relation)

Given T a first-order complete countable theory in a countable vocabulary, we say that $f, g \in \kappa^\kappa$ are $\cong^\kappa_T$ equivalent if

- $\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
  or
- $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

The complexity

We can define a partial order on the set of all first-order complete countable theories

$$T \leqslant_\kappa^\kappa T' \text{ iff } \cong_T^\kappa \leqslant_c^\kappa \cong_{T'}^\kappa$$

The subspace 2^κ

In the subspace 2^κ, we can define the following notions in the same way:

- $E_1 \leqslant_c^2 E_2$.
- $f \cong_T^2 g$.
- $T \leqslant_\kappa^2 T'$.

Outline

Shelah's Main Gap Theorem

Theorem (Shelah)

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Question:
Is there a Borel reducibility counterpart of the Main Gap Theorem in the spaces κ^κ and 2^κ?

$E^{\kappa}_{\lambda\text{-club}}$ and $E^{2}_{\lambda\text{-club}}$

For every regular cardinal $\lambda < \kappa$, the relations $E^{\kappa}_{\lambda\text{-club}}$ and $E^{2}_{\lambda\text{-club}}$ are defined as follow.

Definition

- *On the space κ^κ, we say that $f, g \in \kappa^\kappa$ are $E^{\kappa}_{\lambda\text{-club}}$ equivalent if the set $\{\alpha < \kappa | f(\alpha) = g(\alpha)\}$ contains an unbounded set that is closed under λ-limits.*
- *On the space 2^κ, we say that $f, g \in 2^\kappa$ are $E^{2}_{\lambda\text{-club}}$ equivalent if the set $\{\alpha < \kappa | f(\alpha) = g(\alpha)\}$ contains an unbounded set that is closed under λ-limits.*

Looking above the Gap

Theorem (Friedman, Hyttinen, Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$.

- *If T is an unstable or superstable with OTOP, then $E^2_{\lambda\text{-club}} \leqslant^2_c \cong^2_T$.*
- *If $\lambda \geq 2^\omega$ and T is a superstable with DOP, then $E^2_{\lambda\text{-club}} \leqslant^2_c \cong^2_T$.*

Theorem (Friedman, Hyttinen, Kulikov)

Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and T is a stable unsuperstable. Then $E^2_{\omega\text{-club}} \leqslant^2_c \cong^2_T$

Looking below the Gap

Theorem (Friedman, Hyttinen, Kulikov)

If T is a classifiable theory, then for all regular cardinal $\lambda < \kappa$, $E^2_{\lambda\text{-}club} \not\leqslant^2_c \cong^2_T$

Theorem (Hyttinen, Moreno)

Suppose T is a classifiable theory and $\lambda < \kappa$ is a regular cardinal. Then $\cong^\kappa_T \leqslant^\kappa_c E^\kappa_{\lambda\text{-}club}$.

Theorem (Hyttinen, Kulikov, Moreno)

Denote by S^κ_λ the set $\{\alpha < \kappa | cf(\alpha) = \lambda\}$.
Suppose T is a classifiable theory and $\lambda < \kappa$ is a regular cardinal. If $\diamondsuit(S^\kappa_\lambda)$ holds, then $\cong^2_T \leqslant^2_c E^2_{\lambda\text{-club}}$.

The Gap in ZFC

Theorem (Hyttinen, Moreno)

Suppose T is a classifiable theory, T' is an stable theory with the OCP, and κ an inaccessible cardinal. Then $\cong^\kappa_T \leqslant^\kappa_c \ E^\kappa_{\omega\text{-club}} \leqslant^\kappa_c \cong^\kappa_{T'}$

Theorem (Moreno)

Suppose T is a classifiable theory, T' is a superstable theory with the S-DOP, $\lambda \geq 2^\omega$, and κ an inaccessible cardinal. Then $\cong^\kappa_T \leqslant^\kappa_c \ E^\kappa_{\lambda\text{-club}} \leqslant^\kappa_c \cong^\kappa_{T'}$

Theorem (Hyttinen, Kulikov, Moreno)

Suppose $\kappa = \lambda^+$ and $\lambda^\omega = \lambda$. If T is a classifiable theory and T' is a stable unsuperstable theory, then $\cong^2_T \leqslant^2_c \ E^2_{\omega\text{-club}} \leqslant^2_c \cong^2_{T'}$.

Consistency

Let $H(\kappa)$ be the following property: If T is classifiable and T' is not, then $T \leqslant^2_\kappa T'$ and $T' \nleqslant^2_\kappa T$.

Theorem

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$ *and* $\lambda^{<\lambda} = \lambda$.

1. *If* $V = L$, *then* $H(\kappa)$ *holds.*
2. *It is consistent that* $H(\kappa)$ *holds and there are* 2^κ *equivalence relations strictly between* $\cong^2_{T_1}$ *and* $\cong^2_{T_2}$.

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