The isomorphism relation of theories with S-DOP in generalized Baire spaces

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Abstract

We study the Borel-reducibility of isomorphism relations in the generalized Baire space κ^κ. In the main result we show for inaccessible κ, that if T is a classifiable theory and T' is superstable with the strong dimensional order property (S-DOP), then the isomorphism of models of T is Borel reducible to the isomorphism of models of T'. In fact we show the consistency of the following: If κ is inaccessible and T is a superstable theory with S-DOP, then the isomorphism of models of T is Σ_1^1-complete.

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1 Introduction

One of the main motivations behind the study of the generalized descriptive set theory, is the connections with model theory. The complexity of a theory can be measured using the Borel reducibility in the generalized Baire spaces: We say that T' is more complex than T if the isomorphism relation of T with universe κ ($\cong_T$) is Borel reducible to the isomorphism relation of T' with universe κ. Classification theory in Shelah's stability theory gives another notion of complexity. The stability theory notion of complexity allows us to compare classifiable theories with non-classifiable theories, but it doesn't allows us to compare the complexity of two non-classifiable theories. On the other hand, the Borel reducibility notion of complexity allows us to compare the complexity of two theories, no matter if the theories are both non-classifiable. Friedman, Hyttinen, Kulikov and others have studied the connection between these two notions of complexity.

One of the most important questions regarding the Borel reducibility complexity notion is: *Is the Borel reducibility notion of complexity a refinement of the stability theory notion of complexity?* Answer this question is one of the objective pursued by the generalized descriptive set theory. For a theory to be non-classifiable, this one must be either unstable, or superstable with OTOP, or superstable with DOP, or stable unsuperstable. It is natural for model theorist to believe that there is a distinction between the complexity of these four kind of non-classifiable theories, it is conjectured that this mey be reflected in the Borel reducibility complexity notion.

The results reviewed in this introduction require further assumptions and the reader is referred to the original paper for the exact assumptions. In [HKM] it was shown, under the assumptions of $\diamondsuit$ and κ successor, if T is classifiable and T' is not, then $\cong_T$ is Borel reducible to $\cong_{T'}$. In [Fer], [FMR] and [HKM2] it was showed that for certain models of ZFC, if κ is a successor cardinal, then the isomorphism relations of any non-classifiable theory is Σ_1^1-complete. In particular, in [FMR] and [FMR2] different forcings were constructed to obtain this. It is natural to ask whether the same holds when κ is inaccessible. The case stable unsuperstable was studied in [HM] and the following was found, *if T is classifiable and T' is stable unsuperstable with OCP, then $\cong_T$ is continuously reducible to $\cong_{T'}$*, in some models ([Fer], [FMR], [HKM2]) $\cong_{T'}$ is Σ_1^1-complete. Some previous work has been done in the case of superstable theories with DOP. In [LS] Laskowski and Shelah studied the λ-Borel completeness of the relation $(Mod_\lambda(T), \equiv_{\infty,\aleph_0})$ when T is ω-stable with *eni*-DOP or *eni*-deep (see below).

Definition 1.1. *For any relational language L with size at most λ, let $L^\pm = L \cup \{\neg R \mid R \in L\}$, and let S_L^λ denote the set of L-structures M with universe L. Let $L(\lambda) = \{R(\bar{\alpha}) \mid R \in L^\pm, \bar{\alpha} \in \lambda^n, n = arity(R)\}$ and endow S_L^λ with the topology generated by the subbasis*

$$\mathcal{B} = \{U_{R(\bar{\alpha})} \mid R(\bar{\alpha}) \in L(\lambda)\}$$

where $U_{R(\bar{\alpha})} = \{M \in S_L^\lambda \mid M \models R(\bar{\alpha})\}$.

Definition 1.2. *Given a language L of size at most λ, a set $K \subseteq S_L^\lambda$ is λ-Borel if, there is a λ-Boolean combination ψ of $L(\lambda)$-sentences (i.e., a propositional $L_{\lambda^+,\aleph_0}$-sentence of $L(\lambda)$) such that*

$$K = \{M \in S_L^\lambda \mid M \models \psi\}$$

Given two relational languages L_1 and L_2 of size at most λ, a function $f : S_{L_1}^\lambda \to S_{L_2}^\lambda$ is *λ-Borel* if the inverse image of every open set is λ-Borel.

Definition 1.3. *Suppose that L_1 and L_2 are two relational languages of size at most λ, and for $l = 1, 2$, K_l is a λ-Borel subset of $S_{L_l}^\lambda$ that is invariant under $\equiv_{\infty,\aleph_0}$. We say that $(K_1, \equiv_{\infty,\aleph_0})$ is λ-Borel reducible to $(K_2, \equiv_{\infty,\aleph_0})$, written*

$$(K_1, \equiv_{\infty,\aleph_0}) \leq_\lambda^B (K_2, \equiv_{\infty,\aleph_0})$$

if there is a λ-Borel function $f : S_{L_1}^\lambda \to S_{L_2}^\lambda$ such that $f(K_1) \subseteq K_2$, and for all $M, N \in K_1$ it holds that

$$M \equiv_{\infty,\aleph_0} N \text{ if and only if } f(M) \equiv_{\infty,\aleph_0} f(N)$$

Definition 1.4. *K is λ-Borel complete for $\equiv_{\infty,\aleph_0}$ if $(K,\equiv_{\infty,\aleph_0})$ is a maximum with respect to $\leq^B_\lambda$. We call a theory T λ-Borel complete for $\equiv_{\infty,\aleph_0}$ if $Mod_\lambda(T)$, the class of models of T with universe λ, is λ-Borel complete for $\equiv_{\infty,\aleph_0}$.*

Lemma 1.5 ([LS], Corollary 4.13 and 6.10). *If T is ω-stable with eni-DOP or eni-deep, then T is λ-Borel complete for $\equiv_{\infty,\aleph_0}$*

Let us use the isomorphism relation to make a last observation on the relations $\equiv^K_{\infty,\aleph_0}$. Here and throughout the paper we assume that κ is an uncountable cardinal that satisfies $\kappa^{<\kappa}=\kappa$, $\mathcal{M}$ will denote the monster model, and for every finite tuple a, we will denote $a\in A^{length(a)}$ by $a\in A$, unless something else is stated.

The generalized Baire space is the set κ^κ with the bounded topology. For every $\zeta\in\kappa^{<\kappa}$, the set

$$[\zeta]=\{\eta\in\kappa^\kappa\mid\zeta\subset\eta\}$$

is a basic open set. The open sets are of the form $\bigcup X$ where X is a collection of basic open sets. The collection of Borel subsets of κ^κ is the smallest set which contains the basic open sets and is closed under complement and unions of length κ.

A function $f\colon\kappa^\kappa\to\kappa^\kappa$ is *Borel*, if for every open set $A\subseteq\kappa^\kappa$ the inverse image $f^{-1}[A]$ is a Borel subset of κ^κ. Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function f that satisfies $(x,y)\in E_1\Leftrightarrow(f(x),f(y))\in E_2$, we call f a *reduction* of E_1 to E_2 and it is denoted by $E_1\leq_B E_2$. If f is continuous, then E_1 is *continuously reducible* to E_2 and it is denoted by $E_1\leq_c E_2$.

Let $\mathcal{L}=\{P_n\mid n\in\kappa\backslash\}$ be a given relation vocabulary of size κ. When we describe a complete theory T in a vocabulary $L\subseteq\mathcal{L}$, we think of it as a complete $\mathcal{L}$-theory extending $T\cup\{\forall\bar{x}\neg P_n(\bar{x})\mid P_n\in\mathcal{L}\backslash L\}$. We can code $\mathcal{L}$-structures with domain κ as follows.

Definition 1.6. *Fix a bijection $\pi\colon\kappa^{<\omega}\to\kappa$. For every $\eta\in\kappa^\kappa$ define the $\mathcal{L}$-structure $\mathcal{A}_\eta$ with domain κ as follows: For every relation P_m of arity n, every tuple $(a_1,a_2,\ldots,a_n)$ in κ^n satisfies*

$$(a_1,a_2,\ldots,a_n)\in P_m^{\mathcal{A}_\eta}\iff\eta(\pi(m,a_1,a_2,\ldots,a_n))\geq 1.$$

Since for all $\beta<\kappa$, the sets $\{\eta\in\kappa^\kappa\mid\eta(\beta)=0\}$ and $\{\eta\in\kappa^\kappa\mid\eta(\beta)>0\}$ are Borel, then for all $R\in\mathcal{L}^\pm$ and $\bar{\alpha}\in\kappa^{arity(R)}$ the set $\{\eta\in\kappa^\kappa\mid\mathcal{A}_\eta\models R(\bar{\alpha})\}$ is Borel. Then if K is a κ-Borel subset of $S^\kappa_{\mathcal{L}}$, then the set $\{\eta\in\kappa^\kappa\mid M=\mathcal{A}_\eta,M\in K\}$ is Borel. On the other hand for every basic open set $[\zeta]$, there is φ, a $\mathcal{L}_{\kappa,\aleph_0}$-sentence of $\mathcal{L}(\kappa)$, such that $[\zeta]=\{\eta\in\kappa^\kappa\mid\mathcal{A}_\eta\models\varphi\}$. Therefore, if $K\subseteq S^\kappa_{\mathcal{L}}$ is such that $\{\eta\in\kappa^\kappa\mid M=\mathcal{A}_\eta,M\in K\}$ is Borel, then there is ψ a $\mathcal{L}_{\kappa^+,\aleph_0}$-sentence of $\mathcal{L}(\kappa)$ such that $\{\eta\in\kappa^\kappa\mid M=\mathcal{A}_\eta,M\in K\}=\{\eta\in\kappa^\kappa\mid\mathcal{A}_\eta\models\psi\}$. We conclude that $K\subseteq S^\kappa_{\mathcal{L}}$ is κ-Borel if and only if $\{\eta\in\kappa^\kappa\mid M=\mathcal{A}_\eta,M\in K\}$ is Borel.

Let us define the equivalence relation $\equiv^K_{\infty,\aleph_0}\subset\kappa^\kappa\times\kappa^\kappa$ for every K κ-Borel subset of $S^\kappa_{\mathcal{L}}$ invariant under $\equiv_{\infty,\aleph_0}$ by: $(\eta,\xi)\in\equiv^K_{\infty,\aleph_0}$ if and only if

- $\mathcal{A}_\eta,\mathcal{A}_\xi\in K$ and $\mathcal{A}_\eta\ \equiv_{\infty,\aleph_0}\ \mathcal{A}_\xi$, or
- $\mathcal{A}_\eta,\mathcal{A}_\xi\notin K$.

If $K=Mod_\kappa(T)$, then we denote by $\equiv^T_{\infty,\aleph_0}$ the equivalence relation $\equiv^K_{\infty,\aleph_0}$. From the previous observation, we can restate Lemma 1.5 as follows:

If T is ω-stable with eni-DOP or eni-deep, then for every K κ-Borel subset of $S^\kappa_{\mathcal{L}}$ invariant under $\equiv_{\infty,\aleph_0}$ it holds that

$$\equiv^K_{\infty,\aleph_0}\leq_B\equiv^T_{\infty,\aleph_0}.$$

Definition 1.7 (The isomorphism relation). *Assume T is a complete first order theory in a countable vocabulary, $\mathcal{L}$. We define $\cong^\kappa_T$ as the relation*

$$\{(\eta,\xi)\in\kappa^\kappa\times\kappa^\kappa \mid (\mathcal{A}_\eta\models T,\mathcal{A}_\xi\models T,\mathcal{A}_\eta\cong\mathcal{A}_\xi)\ or\ (\mathcal{A}_\eta\not\models T,\mathcal{A}_\xi\not\models T)\}.$$

We will omit the superscript "κ" in $\cong^\kappa_T$ when it is clear from the context. For every complete first order theory T in a countable vocabulary there is an isomorphism relation associated with T, $\cong^\kappa_T$.

Given a countable vocabulary $\mathcal{L}$, define L by $L=\mathcal{L}\cup\{P\}\cup\{R_\beta\mid\beta<\kappa\}$, where P is an unary relation R_β is a binary relation for all $\beta<\kappa$. Let T be a complete first order theory in $\mathcal{L}$, for every $\mathcal{A}\in Mod_\kappa(T)$ construct an L-structure $\bar{\mathcal{A}}$ such that:

- $dom(\bar{\mathcal{A}})=\kappa$,
- $\bar{\mathcal{A}}\models P(\alpha)$ if and only if there is $\beta<\kappa$ such that $\alpha=2\beta$,
- $\bar{\mathcal{A}}\restriction\{2\beta\mid\beta<\kappa\}$ is isomorphic to $\mathcal{A}$ as an $\mathcal{L}$–structure,
- $\forall\beta<\kappa$, $R_\beta(x,y)$ implies $\neg P(x)\wedge P(y)$,
- for every $\alpha<\kappa$ and every b with $\neg P(b)$, there is a unique tuple $\bar{a}\in\kappa^{<\kappa}$ with $length(\bar{a})=\alpha$ and for all $\gamma<\alpha$, $P(a_\gamma)$, that satisfies:

$$\forall\beta<\alpha,\ R_\beta(b,c)\Leftrightarrow c=a_\beta.$$

- for every $\alpha<\kappa$ and every tuple $\bar{a}\in\kappa^\kappa$ with $length(\bar{a})=\alpha$ and for all $\gamma<\alpha$, $P(a_\gamma)$, there is a unique element of $\bar{\mathcal{A}}$, $b_{\bar{a}}$, that satisfies:

$$\forall\beta<\alpha,\ R_\beta(b_{\bar{a}},c)\Leftrightarrow\neg P(b_a)\ and\ c=a_\beta.$$

Let $\bar{K}$ be the smallest subset of S^κ_L that contains $\{\bar{\mathcal{A}}\mid\mathcal{A}\in K\}$ and is invariant under $\equiv_{\infty,\aleph_0}$. By Theorem XIII.1.4 of [She], if T is a classifiable theory in $\mathcal{L}$, we get that $(\eta,\xi)\in\equiv^T_{\infty,\kappa}$ if and only if $(\eta,\xi)\in\cong_T$. Now, $(\eta,\xi)\in\cong_T$ clearly implies $\bar{\mathcal{A}}_\eta\equiv_{\infty,\aleph_0}\bar{\mathcal{A}}_\xi$; conversely $\bar{\mathcal{A}}_\eta\equiv_{\infty,\aleph_0}\bar{\mathcal{A}}_\xi$ implies $\mathcal{A}_\eta\equiv_{\infty,\kappa}\mathcal{A}_\xi$, so $\bar{\mathcal{A}}_\eta\equiv_{\infty,\aleph_0}\bar{\mathcal{A}}_\xi$ implies $(\eta,\xi)\in\cong_T$. We conclude that the map $f:\kappa^\kappa\to\kappa^\kappa$ given by

- if $\mathcal{A}_\eta\models T$, then $f(\eta)$ is a code for $\bar{\mathcal{A}}_\eta$ (i.e. $\mathcal{A}_{f(\eta)}=\bar{\mathcal{A}}_\eta$),
- if $\mathcal{A}_\eta\not\models T$, then $f(\eta)$ a code for $\mathcal{B}$, where $\mathcal{B}$ is a fix L-structure not in $\bar{K}$.

is a reduction from $\cong_T$ to $\equiv^{\bar{K}}_{\infty,\aleph_0}$. In [FHK] (Theorem 69) it was proved that if T is classifiable and not shallow, then $\cong_T$ is Δ^1_1 and not Borel. Therefore, if T is classifiable and not shallow, then $\equiv^{\bar{K}}_{\infty,\aleph_0}$ is not Borel. In conclusion, for many K κ-Borel subset of $S^\kappa_{\mathcal{L}}$ invariant under $\equiv_{\infty,\aleph_0}$, the relation $\equiv^K_{\infty,\aleph_0}$ is not Borel. Notice that all the relations of the form $\equiv^K_{\infty,\aleph_0}$ are Δ^1_1, this is due to the fact that $\equiv_{\infty,\aleph_0}$ is characterized by the Ehrenfeucht-Fraïssé game of length ω which is a determined game.

From now on $\mathcal{L}$ will be a countable relational vocabulary and every theory is a theory in $\mathcal{L}$. In this paper we study the case of superstable theories with DOP, we answer the question:

Question 1.8. *Is it consistently true: There is a superstable theory with DOP for which the isomorphism relation is Σ^1_1-complete?*

As it was mentioned above, this question was answered when κ is a successor ([FMR], [HKM2]), we will focus only on the case κ an inaccessible cardinal. We answer this question in Corollary 5.3, where we show that in the models of [FMR] and [HKM2], the isomorphism relation of any superstable theory with S-DOP is Σ_1^1-complete. In particular we will prove that there is $\lambda < \kappa$ such that $E^\kappa_{\lambda\text{-club}} \leq_c \cong_T$ holds for any T superstable theory with S-DOP. For every regular cardinal $\mu < \kappa$ and $f, g \in \kappa^\kappa$ are $E^\kappa_{\mu\text{-club}}$ equivalent ($f\ E^\kappa_{\mu\text{-club}}\ g$) if the set $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\}$ contains a μ-club, i.e. it is unbounded and closed under μ-limits.

2 Preliminaries

2.1 Coloured Trees

Coloured trees have been very useful in the past to reduce $E^\kappa_{\mu\text{-club}}$ to $\cong_T$ for certain $\mu < \kappa$ and T non-classifiable, see [FHK], [HM] or [HK]. We will present a variation of these trees that has height $\lambda + 2$ for λ an uncountable cardinal.

For a tree t, for every $x \in t$ we denote by $ht(x)$ the height of x, the order type of $\{y \in t \mid y < x\}$. Define $t_\alpha = \{x \in t \mid ht(x) = \alpha\}$ and $t_{<\alpha} = \cup_{\beta<\alpha} t_\beta$, denote by $x \restriction \alpha$ the unique $y \in t$ such that $y \in t_\alpha$ and $y \leq x$. If $x, y \in t$ and $\{z \in t \mid z < x\} = \{z \in t \mid z < y\}$, then we say that x and y are $\sim$-related, $x \sim y$, and we denote by $[x]$ the equivalence class of x for $\sim$. An α, β-tree is a tree t with the following properties:

- $|[x]| < \alpha$ for every $x \in t$.
- All the branches have order type less than β in t.
- t has a unique root.
- If $x, y \in t$, x and y has no immediate predecessors and $x \sim y$, then $x = y$.

Definition 2.1. *Let λ be an uncountable cardinal. A coloured tree is a pair (t, c), where t is a κ^+, $(\lambda+2)$-tree and c is a map $c : t_\lambda \to \kappa \backslash \{0\}$.*

Two coloured trees (t, c) and (t', c') are isomorphic, if there is a trees isomorphism $f : t \to t'$ such that for every $x \in t_\lambda$, $c(x) = c'(f(x))$. We can see every coloured tree as a downward closed subset of $\kappa^{\leq\lambda}$.

Order the set $\lambda \times \kappa \times \kappa \times \kappa \times \kappa$ lexicographically, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) > (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ if for some $1 \leq k \leq 5$, $\alpha_k > \beta_k$ and for every $i < k$, $\alpha_i = \beta_i$. Order the set $(\lambda \times \kappa \times \kappa \times \kappa \times \kappa)^{\leq\lambda}$ as a tree by inclusion. Define the tree (I_f, d_f) as, I_f the set of all strictly increasing functions from some $\theta \leq \lambda$ to κ and for each η with domain λ, $d_f(\eta) = f(sup(rang(\eta)))$. For every pair of ordinals α and β, $\alpha < \beta < \kappa$ and $i < \lambda$ define

$$R(\alpha, \beta, i) = \bigcup_{i<j\leq\lambda} \{\eta : [i, j) \to [\alpha, \beta) \mid \eta \text{ strictly increasing}\}.$$

Suppose κ is an inaccessible cardinal. If $\alpha < \beta < \kappa$ and $\alpha, \beta, \gamma \neq 0$, let $\{P^{\alpha,\beta}_\gamma \mid \gamma < \kappa\}$ be an enumeration of all downward closed subtrees of $R(\alpha, \beta, i)$ for all i, in such a way that each possible coloured tree appears cofinally often in the enumeration. Let $P^{0,0}_0$ be the tree (I_f, d_f). This enumeration is possible because κ is inaccessible; there are at most $|\bigcup_{i<\lambda} \mathcal{P}(R(\alpha, \beta, i))| \leq \lambda \times \kappa = \kappa$ downward closed coloured subtrees, and at most $\kappa \times \kappa^{<\kappa} = \kappa$ coloured trees. Denote by $Q(P^{\alpha,\beta}_\gamma)$ the unique ordinal number i such that $P^{\alpha,\beta}_\gamma \subset R(\alpha, \beta, i)$.

Definition 2.2. *Suppose κ is an inaccessible cardinal. Define for each $f \in \kappa^\kappa$ the coloured tree (J_f, c_f) by the following construction. For every $f \in \kappa^\kappa$ define $J_f = (J_f, c_f)$ as the tree of all $\eta : s \to \lambda \times \kappa^4$, where $s \leq \lambda$, ordered by extension, and such that the following conditions hold for all $i, j < s$:*

Denote by η_i, $1 \leq i \leq 5$, the functions from s to κ that satisfies, $\eta(n) = (\eta_1(n), \eta_2(n), \eta_3(n), \eta_4(n), \eta_5(n))$.

1. *$\eta \restriction n \in J_f$ for all $n < s$.*
2. *η is strictly increasing with respect to the lexicographical order on $\lambda \times \kappa^4$.*
3. *$\eta_1(i) \leq \eta_1(i+1) \leq \eta_1(i) + 1$.*
4. *$\eta_1(i) = 0$ implies $\eta_2(i) = \eta_3(i) = \eta_4(i) = 0$.*
5. *$\eta_2(i) \geq \eta_3(i)$ implies $\eta_2(i) = 0$.*
6. *$\eta_1(i) < \eta_1(i+1)$ implies $\eta_2(i+1) \geq \eta_3(i) + \eta_4(i)$.*
7. *For every limit ordinal α, $\eta_k(\alpha) = sup_{\beta<\alpha}\{\eta_k(\beta)\}$ for $k \in \{1, 2\}$.*
8. *$\eta_1(i) = \eta_1(j)$ implies $\eta_k(i) = \eta_k(j)$ for $k \in \{2, 3, 4\}$.*
9. *If for some $k < \lambda$, $[i, j) = \eta_1^{-1}\{k\}$, then*

$$\eta_5 \restriction [i, j) \in P^{\eta_2(i), \eta_3(i)}_{\eta_4(i)}.$$

   Note that 7 implies $Q(P^{\eta_2(i), \eta_3(i)}_{\eta_4(i)}) = i$.

10. *If $s = \lambda$, then either*
    (a) *there exists an ordinal number m such that for every $k < m$ $\eta_1(k) < \eta_1(m)$, for every $k' \geq m$ $\eta_1(k) = \eta_1(m)$, and the color of η is determined by $P^{\eta_2(m), \eta_3(m)}_{\eta_4(m)}$:*

    $$c_f(\eta) = c(\eta_5 \restriction [m, \lambda))$$

    where c is the colouring function of $P^{\eta_2(m), \eta_3(m)}_{\eta_4(m)}$.

    Or

    (b) *there is no such ordinal m and then $c_f(\eta) = f(sup(rang(\eta_5)))$.*

The following lemma is a variation of Lemma 4.7 of [HM], nevertheless the proof is the same in both cases.

Lemma 2.3. *Assume κ is an inaccessible cardinal, then for every $f, g \in \kappa^\kappa$ the following holds*

$$f \; E^\kappa_{\lambda\text{-club}} \; g \Leftrightarrow J_f \cong J_g$$

Remark 2.4. *For each $\alpha < \kappa$ define J^α_f as*

$$J^\alpha_f = \{\eta \in J_f \mid rang(\eta) \subset \lambda \times (\beta)^4 \text{ for some } \beta < \alpha\}.$$

Notice that for every $\eta \in J_f$ has the following properties:

1. $sup(rang(\eta_4)) \leq sup(rang(\eta_3)) = sup(rang(\eta_5)) = sup(rang(\eta_2))$.
2. When $\eta \restriction k \in J_f^\alpha$ holds for every $k \in \lambda$, $sup(rang(\eta_5)) \leq \alpha$. If in addition $\eta \notin J_f^\alpha$, then $sup(rang(\eta_5)) = \alpha$.

From now on κ will be an inaccessible cardinal. Let us take a look at the sets $rang(f)$ and $rang(c_f)$, more specifically at the set $\{\alpha < \kappa \mid f(\alpha) \in rang(c_f)\}$.

Remark 2.5. *Assume $f \in \kappa^\kappa$ and let J_f be the respective coloured tree obtained by Definition 2.2. If $\eta \in J_f$ satisfies Definition 2.2 item 10 (b), then clearly exists $\alpha < \kappa$ such that $c_f(\eta) = f(\alpha)$. It is possible that not for every $\alpha < \kappa$, there is $\eta \in J_f^{\alpha+1}$ such that $c_f(\eta) = f(\alpha)$. Nevertheless the set $C = \{\alpha < \kappa \mid \exists \xi \in J_f^{\alpha+1}$ such that $\xi_1 \restriction \omega = id + 1, \xi_1 \restriction [\omega, \lambda) = id \restriction [\omega, \lambda)$ and $c_f(\xi) = f(\alpha)\}$ is a λ-club.*

2.2 Strong DOP

Now, we will recall the dimensional order property and the strong dimensional order property. The independence properties of indiscernible sequences have been a very useful tool to study theories with DOP (see [HaMa], Section 2), this makes superstable theories with DOP and strong independence properties good candidates to answer Question 1.8. Following this direction we will define the strong dimensional property (Lemma 2.9 and Definition 2.13), we will give some important properties that will be useful to construct models of theories with the strong dimensional property. In [She] Shelah gives an axiomatic approach for an isolation notion, F, and defines the notions F-constructible, F-atomic, F-primary, F-prime and F-saturated.

Definition 2.6. *Denote by F_θ^a the set of pairs (p, B) with $|B| < \theta$, such that for some $A \supseteq B$ and a, $p \in S(A)$, $a \models p$ and $stp(a, B) \vdash p$.*

In [She] (Definition II 4.2 (2), and Definition V 1.1 (2) and (4)) the notions of stationarization of a type, and orthogonal types are defined. For $p_1, p_2 \in S(A)$ stationary types the following holds. If $p_1 = tp(a_1, A)$, and $p_2 = tp(a_2, A)$, then p_1 is weakly orthogonal to p_2 if and only if $a_1 \downarrow_A a_2$. A stationary type $p \in S(B)$ is orthogonal to A if for all a, b and $D \supset A$ the following holds: If $tp(b, B)$ is stationary, $a \models p$, $b \downarrow_A B$, $b \downarrow_B D$ and $a \downarrow_B D$, then $a \downarrow_D b$.

Fact 2.7. *Let $B, D \subseteq M$, M a F_ω^a-saturated model over $B \cup D$, and $p \in S(M)$. If p is orthogonal to D and p does not fork over $B \cup D$, then for every $a \models p \restriction B \cup D$ the following holds: $a \downarrow_{B\cup D} M$ implies $tp(a, M) \perp D$.*

A type $p \in S(B \cup C)$ is orthogonal to C, if for every F_ω^a-primary model, M, over $B \cup C$ there exists a non-forking extension of p, $q \in S(M)$, orthogonal to C.

In [She] (X.2 Definition 2.1) Shelah defines the dimensional order property, DOP, as follows.

Definition 2.8. *A theory T has the dimensional order property (DOP) if there are $F_{\kappa(T)}^a$-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, $M_1 \downarrow_{M_0} M_2$, and the $F_{\kappa(T)}^a$-prime model over $M_1 \cup M_2$ is not $F_{\kappa(T)}^a$-minimal over $M_1 \cup M_2$.*

The proof of the following lemma is similar to the proof of [[She] X.2 Lemma 2.2].

Lemma 2.9. *Let $M_0 \subset M_1 \cap M_2$ be F_ω^a-saturated models, $M_1 \downarrow_{M_0} M_2$, M_3 F_ω^a-atomic over $M_1 \cup M_2$ and F_ω^a-saturated. Then the following conditions are equivalent:*

1. *There is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, that does not fork over $M_1 \cup M_2$.*

2. *There is an infinite indiscernible $I \subseteq M_3$ over $M_1 \cup M_2$ that is independent over $M_1 \cup M_2$.*

3. *There is an infinite $I \subseteq M_3$ indiscernible over $M_1 \cup M_2$ and independent over $M_1 \cup M_2$, such that $Av(I, M_3)$ is orthogonal to M_1 and to M_2.*

The rest of the results in this section will be stated and proved for the case of the F^a_ω isolation. Many of those results can be easily generalized to $F^a_{\kappa(T)}$ by making small changes on the proof. From now on we will work only with superstable theories. We know that for every superstable theory T, $\kappa(T) = \omega$.

Lemma 2.10 ([HS], Theorem 2.1). *Let $M_0 \prec M_1, M_2$ be F^a_ω-saturated models, such that $M_1 \downarrow_{M_0} M_2$. Let M_3 be an F^a_ω-prime model over $M_1 \cup M_2$ and let $I \subseteq M_3$ be an indiscernible over $M_1 \cup M_2$ such that $Av(I, M_3)$ is orthogonal to M_1 and to M_2. If $(B_i)_{i<3}$ are sets such that:*

- $B_0 \downarrow_{M_0} M_1 \cup M_2$.
- $B_1 \downarrow_{M_1 \cup B_0} B_2 \cup M_2$.
- $B_2 \downarrow_{M_2 \cup B_0} B_1 \cup M_1$.

Then

$$tp(I, M_1 \cup M_2) \vdash tp(I, M_1 \cup M_2 \cup_{i<3} B_i).$$

The following lemma shows that, if M_1, M_2, and M_3 are models that satisfy Definition 2.8, then we can find models M'_1, M'_2, and M'_3 that extend M_1, M_2, and M_3 respectively and satisfy Definition 2.8.

Lemma 2.11. *Let $M_0 \subset M_1 \cap M_2$ be F^a_ω-saturated models, such that $M_1 \downarrow_{M_0} M_2$ and M_3, the F^a_ω-prime model over $M_1 \cup M_2$, is not F^a_ω-minimal over $M_1 \cup M_2$. If $(M'_i)_{i<3}$ are F^a_ω-saturated models that satisfy:*

- $\forall i < 3$, $M_i \subseteq M'_i$.
- $\forall i < 3$, $M'_i \downarrow_{M_i} M_3$.
- $M'_1 \downarrow_{M'_0} M'_2$.

Then M'_3 the F^a_ω-prime model over $M'_1 \cup M'_2$ is not F^a_ω-minimal over $M'_1 \cup M'_2$.

Remark 2.12. *From the previous lemma we can conclude that if I is independent over $M_1 \cup M_2$, then I is independent over $M'_1 \cup M'_2$.*

Definition 2.13. *We say that a superstable theory T has the strong dimensional order property (S-DOP) if the following holds:*

There are F^a_ω-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, such that $M_1 \downarrow_{M_0} M_2$, and for every M_3 F^a_ω-prime model over $M_1 \cup M_2$, there is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, such that it does not fork over $M_1 \cup M_2$.

From [She] X.2 Lemma 2.2, every superstable theory with S-DOP has DOP. In [HrSo] Hrushovski and Sokolvić proved that the theory of differentially closed fields of characteristic zero (DCF) has eni-DOP, so it has DOP. The reader can find an outline of this proof in [Mar2]. We will show that DFC also has the S-DOP, this can be done following the proof in [Mar2] or the one in [Mar] which uses Rosenlicht's Theorem. We will focus on the proof of the S-DOP property:

There are F^a_ω-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, such that $M_1 \downarrow_{M_0} M_2$, and for every M_3 F^a_ω-prime model over $M_1 \cup M_2$, there is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, such that it does not fork over $M_1 \cup M_2$.

More on DCF (proofs, definitions, references, etc) can be found in [Mar]. Let K be a saturated model of DFC, $k \subseteq K$ and $a \in K^n$, we denote by $k\langle a\rangle$ the differentially closed subfield generated by $k(a)$. If $A \subseteq K$ and for all n, every nonzero $f \in k\{x_1, x_2, \ldots, x_n\}$, and all $a_1, a_2, \ldots, a_n \in A$ it holds that $f(a_1, a_2, \ldots, a_n) \neq 0$, then we say that A is δ-independent over k. For all $k \subseteq K$ denote by k^{dif} the differential closure of k in K.

Theorem 2.14 (Hrushovski, Sokolvić, [Mar] Theorem 7.6, [Mar2] Sections 4, 5). *Suppose K_0 is a differentially closed field with characteristic zero, $\{a, b\}$ is δ-independent over K_0, $K_1 = K_0\langle a\rangle^{dif}$, $K_2 = K_0\langle b\rangle^{dif}$, and $K = K_0\langle a, b\rangle^{dif}$. There is p a type over K that does not fork over $\{a, b\}$ such that $K_1 \downarrow_{K_0} K_2$, $p \perp K_1$, and $p \perp K_2$.*

Corollary 2.15. *DFC has the S-DOP.*

Proof. Let a, b, K_1, K_2, and p be as in Theorem 2.14. By Theorem 2.14 it is enough to show that p does not fork over $K_1 \cup K_2$. This follows since p does not fork over $\{a, b\}$. □

3 Construction of Models

In this section we will use coloured trees to construct models of a superstable theory with S-DOP. To do this, we will need some basic results first and fix some notation. We will study only the superstable theories with S-DOP. Instead of write F^a_ω-constructible, F^a_ω-atomic, F^a_ω-saturated and F^a_ω-saturated we will write a-constructible, a-atomic, a-primary, a-prime and a-saturated. From now on T will be a superstable theory with S-DOP. We will denote by $\lambda(T)$ the least cardinal such that T is λ-stable. Since T is superstable, then $\lambda(T) \leq 2^\omega$, we will denote by λ the cardinal $(2^\omega)^+$.

Definition 3.1. *Let us define the dimension of an indiscernible I over A in M by: $dim(I, A, M) = min\{|J| :$ J is equivalent to I and J is a maximal indiscernible over A in $M\}$. If for all J as above $dim(I, A, M) = |J|$, then we say that the dimension is true.*

Lemma 3.2 ([She]). *If I is a maximal indiscernible set over A in M, then $|I| + \kappa(T) = dim(I, A, M) + \kappa(T)$, and if $dim(I, A, M) \geq \kappa(T)$, then the dimension is true.*

Theorem 3.3 ([She]). *If M is a-primary model over A, and $I \subseteq M$ is an infinite indiscernible set over A, then $dim(I, A, M) = \omega$.*

For any indiscernible sequence $I = \{a_i \mid i < \gamma\}$, we will denote by $I \restriction_\alpha$ the sequence $I = \{a_i \mid i < \alpha\}$. Now for every $f \in \kappa^\kappa$ we will use the the tree J_f given in Definition 2.2, to construct the model $\mathcal{A}^f$. Since T has the S-DOP, by Lemma 2.9 and Lemma 2.10 there are a-saturated models $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of cardinality 2^ω and an indiscernible sequence $\mathcal{I}$ over $\mathcal{B} \cup \mathcal{C}$ of size κ that is independent over $\mathcal{B} \cup \mathcal{C}$ such that

1. $\mathcal{A} \subset \mathcal{B} \cap \mathcal{C}$, $\mathcal{B} \downarrow_{\mathcal{A}} \mathcal{C}$.
2. $Av(\mathcal{I}, \mathcal{B} \cup \mathcal{C})$ is orthogonal to $\mathcal{B}$ and to $\mathcal{C}$.
3. If $(B_i)_{i<3}$ are sets such that:
    (a) $B_0 \downarrow_{\mathcal{A}} \mathcal{B} \cup \mathcal{C}$.
    (b) $B_1 \downarrow_{\mathcal{B} \cup B_0} B_2 \cup \mathcal{C}$.
    (c) $B_2 \downarrow_{\mathcal{C} \cup B_0} B_1 \cup \mathcal{B}$.

Then,

$$tp(\mathcal{I}, \mathcal{B} \cup \mathcal{C}) \vdash tp(\mathcal{I}, \mathcal{B} \cup \mathcal{C} \cup_{i<3} B_i).$$

For every $\xi \in (J_f)_{<\lambda}$ and every $\eta \in (J_f)_\lambda$ (recall t_α at the beginning of the section 2), let $\mathcal{B}_\xi \cong_\mathcal{A} \mathcal{B}$, $\mathcal{A} \preceq \mathcal{B}_\xi$, and $\mathcal{C}_\eta \cong_\mathcal{A} \mathcal{C}$, $\mathcal{A} \preceq \mathcal{C}_\eta$, such that the models $(\mathcal{B}_\xi)_{\xi \in (J_f)_{<\lambda}}$ and $(\mathcal{C}_\eta)_{\eta \in (J_f)_\lambda}$ satisfy the following:

- $\mathcal{B}_\xi \downarrow_\mathcal{A} \bigcup\{\mathcal{B}_\zeta, \mathcal{C}_\theta \mid \zeta \in (J_f)_{<\lambda} \wedge \theta \in (J_f)_\lambda \wedge \zeta \neq \xi\}$.
- $\mathcal{C}_\eta \downarrow_\mathcal{A} \bigcup\{\mathcal{B}_\zeta, \mathcal{C}_\theta \mid \zeta \in (J_f)_{<\lambda} \wedge \theta \in (J_f)_\lambda \wedge \theta \neq \eta\}$.

Notice that all $\xi \in (J_f)_{<\lambda}$ and $\eta \in (J_f)_\lambda$, satisfy

$$\mathcal{B}_\xi \cup \mathcal{C}_\eta \downarrow_\mathcal{A} \bigcup\{\mathcal{B}_\zeta, \mathcal{C}_\theta \mid \zeta \in (J_f)_{<\lambda} \wedge \theta \in (J_f)_\lambda \wedge \zeta \neq \xi \wedge \theta \neq \eta\}.$$

Let $F_{\xi\eta}$ be an automorphism of the monster model such that $F_{\xi\eta} \restriction \mathcal{C} : \mathcal{C} \to \mathcal{C}_\eta$ and $F_{\xi\eta} \restriction \mathcal{B} : \mathcal{B} \to \mathcal{B}_\xi$ are isomorphisms and $F_{\xi\eta} \restriction \mathcal{A} = id$. Denote the sequence $\mathcal{I}$ by $\{w_\alpha \mid \alpha < \kappa\}$. For all $\eta \in (J_f)_\lambda$ and every $\xi < \eta$, let $I_{\xi\eta} = \{b_\alpha \mid \alpha < c_f(\eta)\}$ be an indiscernible sequence over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$ of size $c_f(\eta)$, that is independent over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$, that satisfies:

- $tp(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta) = tp(F_{\xi\eta}(\mathcal{I} \restriction c_f(\eta)), \mathcal{B}_\xi \cup \mathcal{C}_\eta)$.
- $I_{\xi\eta} \downarrow_{\mathcal{B}_\xi \cup \mathcal{C}_\eta} \bigcup\{\mathcal{B}_\zeta, \mathcal{C}_\theta \mid \zeta \in (J_f)_{<\lambda} \wedge \theta \in (J_f)_\lambda\} \cup \bigcup\{I_{\zeta\theta} \mid \zeta \neq \xi \vee \theta \neq \eta\}$.

Therefore, there is an elementary embedding $G : \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup F_{\xi\eta}(\mathcal{I} \restriction c_f(\eta)) \to \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup I_{\xi\eta}$ given by $G \restriction \mathcal{B}_\xi \cup \mathcal{C}_\eta = id$ and $G(F_{\xi\eta}(\mathcal{I} \restriction c_f(\eta))) = I_{\xi\eta}$. So the map $H_{\xi\eta} : \mathcal{B} \cup \mathcal{C} \cup \mathcal{I} \restriction c_f(\eta) \to \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup I_{\xi\eta}$ given by $H_{\xi\eta} = G \circ (F_{\xi\eta} \restriction dom(H_{\xi\eta}))$ is elementary.

Remark 3.4. *$\mathcal{B}_\xi$, $\mathcal{C}_\eta$, and $I_{\xi\eta}$ satisfy the following:*

1. *$Av(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta)$ is orthogonal to $\mathcal{B}_\xi$ and to $\mathcal{C}_\eta$.*
2. *If $(B_i)_{i<3}$ are sets such that:*
   (a) $B_0 \downarrow_\mathcal{A} \mathcal{B}_\xi \cup \mathcal{C}_\eta$.
   (b) $B_1 \downarrow_{\mathcal{B}_\xi \cup B_0} B_2 \cup \mathcal{C}_\eta$.
   (c) $B_2 \downarrow_{\mathcal{C}_\eta \cup B_0} B_1 \cup \mathcal{B}_\xi$.

   Then,
   $$tp(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta) \vdash tp(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup_{i<3} B_i).$$
3. $I_{\xi\eta} \downarrow_{\mathcal{B}_\xi \cup \mathcal{C}_\eta} \bigcup\{\mathcal{B}_\zeta, \mathcal{C}_\theta \mid \zeta \in (J_f)_{<\lambda} \wedge \theta \in (J_f)_\lambda\} \cup \bigcup\{I_{\zeta\theta} \mid \zeta \neq \xi \vee \theta \neq \eta\}$.

Definition 3.5. *Let Γ_f be the set $\bigcup\{\mathcal{B}_\xi, \mathcal{C}_\eta, I_{\xi\eta} \mid \xi \in (J_f)_{<\lambda} \wedge \eta \in (J_f)_\lambda \wedge \xi < \eta\}$ and let $\mathcal{A}^f$ be the a-primary model over Γ_f. Let Γ_f^α be the set $\bigcup\{\mathcal{B}_\xi, \mathcal{C}_\eta, I_{\xi\eta} \mid \xi, \eta \in J_f^\alpha \wedge \xi < \eta\}$, recall J_f^α from Remark 2.4.*

Fact 3.6. *If α is such that $\alpha^\lambda < f(\alpha)$, $sup(\{c_f(\eta)\}_{\eta \in J_f^\alpha}) < \alpha$, then $|\Gamma_f^{\alpha+1}| = f(\alpha)$.*

Proof. Since $\Gamma_f^\alpha = \cup\{\mathcal{B}_\xi, \mathcal{C}_\eta, I_{\xi\eta} \mid \xi \in (J_f^\alpha)_{<\lambda} \wedge \eta \in (J_f^\alpha)_\lambda \wedge \xi < \eta\}$, we know that $|\Gamma_f^{\alpha+1}| \leq |J_f^{\alpha+1}| \cdot sup(\{c_f(\eta)\}_{\eta \in (J_f^{\alpha+1})_\lambda})$. Since $|J_f^{\alpha+1}| \leq \alpha^\lambda < f(\alpha)$ and $sup(\{c_f(\eta)\}_{\eta \in J_f^\alpha}) < \alpha < f(\alpha)$, we get $|\Gamma_f^{\alpha+1}| \leq max(f(\alpha), sup(\{c_f(\eta)\}_{\eta \in J_f^{\alpha+1} \setminus J_f^\alpha}))$. But every $\eta \in J_f^{\alpha+1} \setminus J_f^\alpha$ with domain λ has $rang(\eta_1) = \lambda$ and $f(\alpha) = c_f(\eta)$, otherwise $rang(\eta_5) < \alpha$ and $\eta \in J_f^\alpha$. We conclude $|\Gamma_f^{\alpha+1}| = f(\alpha)$. □

Lemma 3.7. *For every $\xi \in (J_f)_{<\lambda}$, $\eta \in (J_f)_\lambda$, $\xi < \eta$, let $p_{\xi\eta}$ be the type $Av(I_{\xi\eta} \restriction \omega, I_{\xi\eta} \restriction \omega \cup \mathcal{B}_\xi \cup \mathcal{C}_\eta)$. If $c_f(\eta) > \omega$, then $dim(p_{\xi\eta}, \mathcal{A}^f) = c_f(\eta)$.*

Proof. Denote by S the set $I_{\xi\eta} \restriction \omega \cup \mathcal{B}_\xi \cup \mathcal{C}_\eta$, so $p_{\xi\eta} = Av(I_{\xi\eta} \restriction \omega, S)$.

Suppose, towards a contradiction, that $dim(p_{\xi\eta}, \mathcal{A}^f) \neq c_f(\eta)$. Since $I_{\xi\eta} \subset \mathcal{A}^f$, then $dim(p_{\xi\eta}, \mathcal{A}^f) > c_f(\eta)$. Therefore, there is an independent sequence $I = \{a_i \mid i < c_f(\eta)^+\}$ over S such that $I \subset \mathcal{A}^f$ and $\forall a \in I$, $a \models p_{\xi\eta}$.

By induction on α, it can be proved that $I_{\xi\eta} \restriction \omega \cup \{a_i \mid i \leq \alpha\}$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$. Therefore $I_{\xi\eta} \restriction \omega \cup I$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$. In particular $I_{\xi\eta} \restriction \omega \cup I$ is indiscernible, and $I_{\xi\eta}$ is equivalent to I.

By some forking calculus manipulation and Remark 3.4, it is easy to prove that $tp(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta) \vdash tp(I_{\xi\eta}, \Gamma_f \backslash I_{\xi\eta})$ and $I_{\xi\eta}$ is indiscernible over $\Gamma_f \backslash I_{\xi\eta}$.

We know that $tp(I, \mathcal{B}_\xi \cup \mathcal{C}_\eta) = tp(I_{\xi\eta}, \mathcal{B}_\xi \cup \mathcal{C}_\eta)$, therefore $tp(I, \mathcal{B}_\xi \cup \mathcal{C}_\eta) \vdash tp(I_{\xi\eta}, \Gamma_f \backslash I_{\xi\eta})$. We conclude that $tp(I, \mathcal{B}_\xi \cup \mathcal{C}_\eta) \vdash tp(I, \Gamma_f \backslash I_{\xi\eta})$ and since I is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$, then I is indiscernible over $\Gamma_f \backslash I_{\xi\eta}$.

There are $I', I^* \subseteq I$ such that $|I'| = c_f(\eta)^+$ and $I' \downarrow_{(\Gamma_f \backslash I_{\xi\eta}) \cup I^*} I_{\xi\eta}$. In particular I' is indiscernible over $\Gamma_f \cup I^*$, and I' is indiscernible over Γ_f.

Let $J \subset \mathcal{A}^f$ be a maximal indiscernible set over Γ_f such that $I' \subseteq J$. By Lemma 3.2 $|J| + \kappa(T) = dim(J, \Gamma_f, \mathcal{A}^f) + \kappa(T)$. Since T is superstable, $\kappa(T) < \omega < c_f(\eta)^+ < |J|$ and we conclude that $\kappa(T) < dim(J, \Gamma_f, \mathcal{A}^f) + \kappa(T)$. Therefore $\kappa(T) < dim(J, \Gamma_f, \mathcal{A}^f)$ and by Lemma 3.2 the dimension is true, $dim(J, \Gamma_f, \mathcal{A}^f) = |J|$. So $dim(J, \Gamma_f, \mathcal{A}^f) > \omega$ a contradiction with Theorem 3.3. □

One of the key lemmas for the following section is Lemma 3.9 (below). Let us define the nice subsets of Γ_f. These subsets have a couple of properties, that will be useful when we study the model $\mathcal{A}^f$.

Definition 3.8. *We say $X \subseteq \Gamma_f$ is nice if the following holds.*

1. *If $X \cap I_{\xi\eta} \neq \emptyset$, then $\mathcal{B}_\xi, \mathcal{C}_\eta \subset X$.*
2. *If $\mathcal{B}_\xi \cap X \neq \emptyset$, then $\mathcal{B}_\xi \subset X$.*
3. *If $\mathcal{C}_\eta \cap X \neq \emptyset$, then $\mathcal{C}_\eta \subset X$.*
4. *If $\xi < \eta$ and $\mathcal{B}_\xi, \mathcal{C}_\eta \subset X$, then $X \cap I_{\xi\eta}$ is infinite.*

The argument for the following Lemma is a variation of the argument used in [HS] in the fourth section.

Lemma 3.9. *Let Z be a nice subset of Γ_f and $d \in \Gamma_f \backslash Z$. Then for all B finite subset of Z there is $f \in Saut(\mathcal{M}, B)$ such that $f(d) \in Z$.*

Suppose X and A are nice subsets of Γ_f. If ξ and η are such that $\mathcal{B}_\xi \cup \mathcal{C}_\eta \subseteq A$ and $I_{\xi\eta} \cap X \subseteq A$, then we say that A is X-nice for (ξ, η).

Lemma 3.10. *Suppose $Z \subseteq \Gamma_f$ is nice and B is a-constructable over Z. If $X \subseteq \Gamma_f$ is a nice subset such that $Z \cup X$ is nice, then $B \cup X$ is a-constructible over $Z \cup X$.*

Proof. Let $(Z, (a_i, B_i)_{i<\gamma})$ be an a-construction for B over Z. Let $(\mathcal{D}_i)_{i<\delta}$ be an enumeration of $\{\mathcal{B}_\xi, \mathcal{C}_\eta, I_{\xi\eta} \cap X \mid \xi < \eta \wedge \mathcal{B}_\xi \cup \mathcal{C}_\eta \subseteq Z \cup X\}$ such that $\mathcal{B}_\xi$ and $\mathcal{C}_\eta$ are before $I_{\xi\eta}$ in the enumeration. Let Z^j be the minimal nice subset of $Z \cup X$ that contains $Z \cup \bigcup_{i \leq j} \mathcal{D}_i$, and it is X-nice for every (x, y) that satisfies: either $\mathcal{B}_x \subseteq \bigcup_{i \leq j} \mathcal{D}_i \backslash Z$ or $\mathcal{C}_y \subseteq \bigcup_{i \leq j} \mathcal{D}_i \backslash Z$. First, we will show that $(Z^j, (a_i, B_i)_{i<\gamma})$ is an a-construction for $B \cup Z^j$ over Z^j, for every $j < \delta$.

Suppose, towards a contradiction, that α is the minimal ordinal such that $(Z^\alpha, (a_i, B_i)_{i<\gamma})$ is not an a-construction for $B \cup Z^\alpha$ over Z^α. By the minimality of α, $(Z^\beta, (a_i, B_i)_{i<\gamma})$ is an a-construction for $B \cup Z^\beta$ over Z^β, for every $\beta < \alpha$. Therefore for every $\beta < \alpha$ and $i < \gamma$, $(tp(a_i, Z_i^\beta), B_i) \in F_\omega^a$ where $Z_i^\beta = Z^\beta \cup \bigcup_{j<i} a_j$. So $(tp(a_i, \cup_{\beta<\alpha} Z_i^\beta), B_i) \in F_\omega^a$ for every $i < \gamma$, we conclude that α is not a limit cardinal. Let us denote by Z' the set Z^β, for β the predecessor of α.

The proof is divided in the following cases:

1. $\mathcal{D}_\alpha = \mathcal{C}_\eta$ for some $\mathcal{C}_\eta \subseteq X \cup Z$.

2. $\mathcal{D}_\alpha = \mathcal{B}_\xi$ for some $\mathcal{B}_\xi \subseteq X \cup Z$.

3. $\mathcal{D}_\alpha = I_{\xi\eta} \cap X$, for some $\mathcal{B}_\xi \cup \mathcal{C}_\eta \subseteq X \cup Z$.

The case 2 is similar to the case 1, we will show only the cases 1 and 3.

Case 1: Since $(Z^\alpha, (a_i, B_i)_{i<\gamma})$ is not an a-construction over Z^α, then by the minimality of Z^α, $\mathcal{C}_\eta \not\subseteq Z'$. Therefore, $I_{\xi\eta} \cap Z' = \emptyset$ for every $\xi < \eta$. Since $X \cup Z$ is nice, then we know that for all $\mathcal{B}_\xi \subseteq Z'$ that satisfies $\xi < \eta$, it holds that $\mathcal{B}_\xi \subseteq X$. Let n be the least ordinal such that $(Z' \cup \mathcal{C}_\eta \cup \bigcup\{I_{\xi\eta} \cap X \mid \xi < \eta \wedge \mathcal{B}_\xi \subseteq Z'\}, (a_i, B_i)_{i \leq n})$ is not an a-construction over $Z' \cup \mathcal{C}_\eta \cup \bigcup\{I_{\xi\eta} \cap X \mid \xi < \eta \wedge \mathcal{B}_\xi \subseteq Z'\}$, since a-isolation is the F_ω^a-isolation, then B_n is finite and we can assume $n < \omega$.

Denote by D the set $\mathcal{C}_\eta \cup \bigcup\{I_{\xi\eta} \cap X \mid \xi < \eta \wedge \mathcal{B}_\xi \subseteq Z'\}$. Since $(Z' \cup D, (a_i, B_i)_{i<n})$ is an a-construction over Z', then $C = \bigcup_{i<n} B_i \cap (Z' \cup D)$ is such that $stp(a_0^\frown \cdots ^\frown a_{n-1}, C) \vdash tp(a_0^\frown \cdots ^\frown a_{n-1}, Z' \cup D)$. Notice that C is a subset of Z'. On the other hand, there is b such that $stp(b, B_n) = stp(a_n, B_n)$, and $tp(b, Z' \cup \bigcup\{a_i \mid i < n\} \cup D) \neq tp(a_n, Z' \cup \bigcup\{a_i \mid i < n\} \cup D)$. So there are tuples $d \in D \backslash \mathcal{A}$ and $e \in Z' \cup \bigcup\{a_i \mid i < n\}$ that satisfy $tp(b, e \cup d) \neq tp(a_n, e \cup d)$. Denote by W the set $C \cup ((B_n \cup e) \cap Z')$, by Lemma 3.9 we know that there is $g \in Saut(\mathcal{M}, W)$ such that $g(d) \in Z'$. We know that, $stp(a_0^\frown \cdots ^\frown a_{n-1}, C) \vdash tp(a_0^\frown \cdots ^\frown a_{n-1}, Z' \cup D)$, so $a_0^\frown \cdots ^\frown a_{n-1} \downarrow_C Z' \cup D$. We conclude that

$$a_0^\frown \cdots ^\frown a_{n-1} \downarrow_W d$$

and

$$a_0^\frown \cdots ^\frown a_{n-1} \downarrow_W g(d).$$

Therefore $stp(d, C \cup B_n \cup e) = stp(g(d), \cup C \cup B_n \cup e)$ and there is $f \in Saut(\mathcal{M}, C \cup B_n \cup e)$ that satisfies $f(d) = g(d)$.

Since $tp(b, e \cup d) \neq tp(a_n, e \cup d)$ and $stp(b, B_n) = stp(a_n, B_n)$ hold, then we have that $tp(f(b), e \cup f(d)) \neq tp(f(a_n), e \cup f(d))$, and the strong types of $a_n, b, f(a_n)$ and $f(b)$ over B_n are the same strong type. Since $(Z', (a_i, B_i)_{i<\gamma})$ is an a-construction, then by the a-isolation we know that $stp(a, B_n) \vdash tp(a_n, Z' \cup \bigcup\{a_i \mid i < n\})$, on the other hand $stp(a_n, B_n) = stp(f(a_n), B_n) = stp(f(b), B_n)$, so $tp(f(a_n), Z' \cup \bigcup\{a_i \mid i < n\}) = tp(f(b), Z' \cup \bigcup\{a_i \mid i < n\})$. In particular $e, f(d) \in Z'$, so $tp(f(b), e \cup f(d)) = tp(f(a_n), e \cup f(d))$, a contradiction.

Case 3: By the way $(\mathcal{D}_i)_{i<\delta}$ was define, we know that $\mathcal{B}_\xi$ and $\mathcal{C}_\eta$ are before $I_{\xi\eta} \cap X$ in the enumeration, so $\mathcal{B}_\xi \cup \mathcal{C}_\xi \subseteq Z'$. We have the following possibilities, either $\mathcal{B}_\xi \not\subseteq Z$, or $\mathcal{C}_\eta \not\subseteq Z$, or $\mathcal{B}_\xi, \mathcal{C}_\eta \subseteq Z$. In the first two cases, by the way Z' was defined, we know that Z' is X-nice for (ξ, η), so $I_{\xi\eta} \cap X \subset Z'$. Therefore, $Z' = Z^\alpha$ and $(Z', (a_i, B_i)_{i<\gamma})$ is an a-construction for $B \cup Z^\alpha$ over Z^α, a contradiction. Therefore, we need to show only the case when $\mathcal{B}_\xi, \mathcal{C}_\eta \subset Z$. Since $(Z^\alpha, (a_i, B_i)_{i<\gamma})$ is not an a-construction over Z^α, then $I_{\xi\eta} \cap X \not\subseteq Z'$.

Let n be the least ordinal such that $(Z' \cup (I_{\xi\eta} \cap X), (a_i, B_i)_{i\leq n})$ is not an a-construction over $Z' \cup (I_{\xi\eta} \cap X)$, since a-isolation is the F^a_ω-isolation, then B_n is finite and we can assume $n < \omega$. Since $(Z' \cup (I_{\xi\eta} \cap X), (a_i, B_i)_{i<n})$ is an a-construction over $Z' \cup (I_{\xi\eta} \cap X)$, then $C = \bigcup_{i<n} B_i \cap (Z' \cup (I_{\xi\eta} \cap X))$ is such that $stp(a_0^\frown \cdots ^\frown a_{n-1}, C) \vdash tp(a_0^\frown \cdots ^\frown a_{n-1}, Z' \cup (I_{\xi\eta} \cap X))$. Notice that C is a subset of Z'. On the other hand, there is b such that $stp(b, B_n) = stp(a_n, B_n)$, and $tp(b, Z' \cup \bigcup\{a_i \mid i < n\} \cup (I_{\xi\eta} \cap X)) \neq tp(a_n, Z' \cup \bigcup\{a_i \mid i < n\} \cup (I_{\xi\eta} \cap X))$. Since Z' is nice, then there is an infinite $I'_{\xi\eta} \subset I_{\xi\eta} \cap X$ contained in Z'. Therefore, there are tuples $d \in (I_{\xi\eta} \cap X) \backslash I'_{\xi\eta}$ and $e \in Z' \cup \bigcup\{a_i \mid i < n\}$ that satisfy $tp(b, e \cup d) \neq tp(a_n, e \cup d)$. Denote by W the set $C \cup ((B_n \cup e) \cap Z')$, by Lemma 3.9 we know that there is $g \in Saut(\mathcal{M}, W)$ such that $g(d) \in Z'$. Since $stp(a_0^\frown \cdots ^\frown a_{n-1}, C) \vdash tp(a_0^\frown \cdots ^\frown a_{n-1}, Z' \cup (I_{\xi\eta} \cap X))$, then $a_0^\frown \cdots ^\frown a_{n-1} \downarrow_C Z' \cup (I_{\xi\eta} \cap X)$. Therefore

$$a_0^\frown \cdots ^\frown a_{n-1} \downarrow_W d$$

and

$$a_0^\frown \cdots ^\frown a_{n-1} \downarrow_W g(d).$$

So, $stp(d, C \cup B_n \cup e) = stp(g(d), \cup C \cup B_n \cup e)$ and there is $f \in Saut(\mathcal{M}, C \cup B_n \cup e)$ that satisfies $f(d) = g(d)$.

Since $tp(b, e \cup d) \neq tp(a_n, e \cup d)$ and $stp(b, B_n) = stp(a_n, B_n)$ hold, we have that $tp(f(b), e \cup f(d)) \neq tp(f(a_n), e \cup f(d))$, and $a_n, b, f(a_n)$ and $f(b)$ have the same strong type over B_n. Since $(Z', (a_i, B_i)_{i<\gamma})$ is an a-construction, then by the a-isolation we know that $stp(a, B_n) \vdash tp(a_n, Z' \cup \bigcup\{a_i \mid i < n\})$, on the other hand $stp(a_n, B_n) = stp(f(a_n), B_n) = stp(f(b), B_n)$, so $tp(f(a_n), Z' \cup \bigcup\{a_i \mid i < n\}) = tp(f(b), Z' \cup \bigcup\{a_i \mid i < n\})$. In particular $e, f(d) \in Z'$, so $tp(f(b), e \cup f(d)) = tp(f(a_n), e \cup f(d))$, a contradiction.

Finally, since for every $\beta < \delta$ and $i < \gamma$, $(tp(a_i, Z_i^\beta), B_i) \in F^a_\omega$ where $Z_i^\beta = Z^\beta \cup \bigcup_{j<i} a_j$, then $(tp(a_i, \cup_{\beta<\delta} Z_i^\beta), B_i) \in F^a_\omega$ and $(\Gamma_f, (a_i, B_i)_{i<\gamma})$ is an a-construction for $B \cup \Gamma_f$ over Γ_f. □

Fact 3.11. *If $Z \subseteq \Gamma_f$ is nice, then for every $\alpha < \kappa$ the following holds*

$$Z \downarrow_{Z \cap \Gamma_f^\alpha} \Gamma_f^\alpha.$$

Corollary 3.12. *If $Z \subseteq \Gamma_f$ is nice, then for every nice set $\Gamma \subseteq \Gamma_f$ the following holds*

$$Z \downarrow_{Z \cap \Gamma} \Gamma.$$

Now, we have all the tools needed to prove the main result of $\mathcal{A}^f$.

4 Main result on $\mathcal{A}^f$

This section is devoted to prove, for certain kind of functions, that the models $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic if and only if J_f and J_g are isomorphic coloured trees.

Theorem 4.1. *Assume f, g are functions from κ to $Card \cap \kappa \backslash \lambda$ such that $f(\alpha), g(\alpha) > \alpha^{++}$ and $f(\alpha), g(\alpha) > \alpha^\lambda$. Then $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic if and only if f and g are $E^\kappa_{\lambda\text{-club}}$ equivalent.*

Lemma 4.2. *Assume f, g are functions from κ to $Card \cap \kappa \backslash \lambda$ such that $f(\alpha), g(\alpha) > \alpha^{++}$ and $f(\alpha), g(\alpha) > \alpha^\lambda$. If f and g are $E^\kappa_{\lambda\text{-club}}$ equivalent, then $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic.*

Proof. Assume f and g are $E^{\kappa}_{\lambda\text{-club}}$ equivalent. By Lemma 2.3 J_f and J_g are isomorphic coloured trees, let $G : J_f \to J_g$ be an isomorphism. Define $\mathcal{H}_{\xi\eta} : \mathcal{B}_{\xi} \cup \mathcal{C}_{\eta} \cup I_{\xi\eta} \to \mathcal{B}_{G(\xi)} \cup \mathcal{C}_{G(\eta)} \cup I_{G(\xi)G(\eta)}$ by $\mathcal{H}_{\xi\eta} = H_{G(\xi)G(\eta)} \circ H^{-1}_{\xi\eta}$ (where H_{rp} is the elementary embedding used in the construction of I_{rp}), clearly $\mathcal{H}_{\xi\eta}$ is elementary. It is easy to check that the map

$$\mathcal{H} = \bigcup_{\eta \in (J_f)_{\lambda}} \bigcup_{\xi \in (J_f)_{<\lambda}, \xi < \eta} \mathcal{H}_{\xi\eta}$$

is elementary. Let $\bar{\mathcal{H}}$ be an automorphism that extends $\mathcal{H}$, then $\bar{\mathcal{H}}(\mathcal{A}^f)$ is a-primary over Γ_g. Therefore $\bar{\mathcal{H}}(\mathcal{A}^f)$ and $\mathcal{A}^g$ are isomorphic, we conclude that $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic.

Lemma 4.3. *Assume f, g are functions from κ to $Card \cap \kappa \backslash \lambda$ such that $f(\alpha), g(\alpha) > \alpha^{++}$ and $f(\alpha), g(\alpha) > \alpha^{\lambda}$. If $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic, then f and g are $E^{\kappa}_{\lambda\text{-club}}$ equivalent.*

Proof. Let us assume, towards a contradiction, that f and g are not $E^{\kappa}_{\lambda\text{-club}}$ equivalent and there is an isomorphism $\Pi : \mathcal{A}^f \to \mathcal{A}^g$. Without loss of generality, we can assume that $\{\alpha \mid f(\alpha) > g(\alpha) \wedge cf(\alpha) = \lambda\}$ is stationary. Let $(\Gamma_f, (a^f_i, B^f_i)_{i<\gamma})$ be an a-construction of $\mathcal{A}^f$ over Γ_f. For every α define $\mathcal{A}^{\alpha}_f = \Gamma^{\alpha}_f \cup \bigcup\{a^f_i \mid i < \alpha\}$, clearly $\mathcal{A}^{\alpha}_f$ is not necessary a model.

We say that $\alpha < \kappa$ is f-good if $(\Gamma^{\alpha}_f, (a^f_i, B^f_i)_{i<\alpha})$ is an a-construction over Γ^{α}_f, $\mathcal{A}^{\alpha}_f$ is an a-primary model over Γ^{α}_f, and α is a cardinal. Notice that there are club many f-good cardinals. We say that α is very good if, α is f-good, $f(\alpha) > g(\alpha) > \alpha^{++}$ and $\Pi(\mathcal{A}^{\alpha}_f) = \mathcal{A}^{\alpha}_g$. Notice that since there are club many α's satisfying $\pi(\mathcal{A}^{\alpha}_f) = \mathcal{A}^{\alpha}_g$ and stationary many α's with cofinality λ such that $f(\alpha) > g(\alpha)$, there are stationary many very good cardinals. Since there are club many α's satisfying $sup(\{c_g(p)\}_{p \in J^{\alpha}_g}) < \alpha$, by Remark 2.5 we can choose α a very good cardinal with cofinality λ and $\eta \in J_f$, such that the following holds:

- $\alpha^{\lambda} < g(\alpha)$,
- $sup(\{c_g(p)\}_{p \in J^{\alpha}_g}) < \alpha$,
- there are cofinally many very good cardinals $\beta < \alpha$,
- $\bigcup rang(\eta_1) = \lambda$ and $\bigcup rang(\eta_5) = \alpha$.

Notice that by Definition 2.2 item 10, $c_f(\eta) = f(\alpha)$. Let us choose $X \subseteq \Gamma_g$ and $Y \subseteq \gamma$ such that:

- Y has power 2^{ω} and is closed (i.e. for all $i \in Y$, $B^g_i \subseteq \Gamma_g \cup \bigcup_{j \in Y} a^g_j$).
- X has power 2^{ω} and is nice.
- $D = X \cup \bigcup\{a^g_i \mid i \in Y\}$ is the a-primary model over X.
- $D^{\alpha} = (X \cap \Gamma^{\alpha}_g) \cup \bigcup\{a^g_i \mid i \in Y \wedge i < \alpha\}$ is the a-primary model over $X \cap \Gamma^{\alpha}_g$.
- $\Pi(\mathcal{C}_{\eta}) \subseteq D$ and $\Pi(\mathcal{A}) \subseteq D^{\alpha}$.
- If $\xi \in (J_g)_{<\lambda}$ is such that $\mathcal{B}_{\xi} \subseteq X$, then for all $\zeta < \xi$, $\mathcal{B}_{\zeta} \subseteq X$.

- If $\theta \in (J_g)_\lambda \backslash J_g^{\alpha+1}$ is such that $\mathcal{C}_\theta \subseteq X$, then for all $\zeta \in J_g^\alpha$, $\zeta < \theta$ implies that $\mathcal{B}_\zeta \subseteq X$.

Notice that since $D = X \cup \bigcup\{a_i^g \mid i \in Y\}$ is an a-construction over X, then for all $i \in Y$, $B_i^g \subseteq X \cup \bigcup_{j\in Y} a_j^g$ holds. Let E be an a-primary model over $\Gamma_g^{\alpha+1} \cup \mathcal{A}_g^\alpha \cup D$. By the definition of $\mathcal{A}^g$, we know that $stp(a_i^g, B_i^g) \vdash tp(a_i^g, \Gamma_g \cup \bigcup\{a_j^g \mid j < i\})$. Since $B_i^g \subseteq X \cup \bigcup\{a_j^g \mid j < i \wedge j \in Y\}$ holds for every $i \in Y$, then $stp(a_i^g, B_i^g) \vdash tp(a_i^g, X \cup \Gamma_g^\alpha \cup \bigcup\{a_j^g \mid j < \alpha\} \cup \bigcup\{a_j^g \mid j < i \wedge j \in Y\})$ holds for all $i \in Y\backslash\alpha$. We conclude that $D \cup \mathcal{A}_g^\alpha$ is a-constructable over $X \cup \mathcal{A}_g^\alpha$. Notice that $X \cup \Gamma_g^\alpha$ is nice, so by Lemma 3.10 $X \cup \mathcal{A}_g^\alpha$ is a-constructable over $X \cup \Gamma_g^\alpha$. We conclude by Lemma 3.10 that E is a-constructable over $\Gamma_g^{\alpha+1} \cup X$. Let F be an a-primary model over $E \cup \bigcup\{\mathcal{B}_\xi, I_{\xi\theta} \mid \xi < \theta \wedge \mathcal{C}_\theta \subseteq X\backslash\Gamma_g^{\alpha+1}\}$, notice that $\Gamma_g^{\alpha+1} \cup X \cup \bigcup\{\mathcal{B}_\xi, I_{\xi\theta} \mid \xi < \theta \wedge \mathcal{C}_\theta \subseteq X\backslash\Gamma_g^{\alpha+1}\}$ is nice and by Lemma 3.10 we conclude that F is a-constructable over $\Gamma_g^{\alpha+1} \cup X \cup \bigcup\{\mathcal{B}_\xi, I_{\xi\theta} \mid \xi < \theta \wedge \mathcal{C}_\theta \subseteq X\backslash\Gamma_g^{\alpha+1}\}$. Let G be an a-primary model over $\Gamma_g \cup F$, since F is a-constructable over $\Gamma_g^{\alpha+1} \cup X \cup \bigcup\{\mathcal{B}_\xi, I_{\xi\theta} \mid \xi < \theta \wedge \mathcal{C}_\theta \subseteq X\backslash\Gamma_g^{\alpha+1}\}$, then by Lemma 3.10 G is a-primary over $\Gamma_g^{\alpha+1} \cup X \cup \bigcup\{\mathcal{B}_\xi, I_{\xi\theta} \mid \xi < \theta \wedge \mathcal{C}_\theta \subseteq X\backslash\Gamma_g^{\alpha+1}\} \cup \Gamma_g$. Without loss of generality, we can assume $G = \mathcal{A}^g$.

Since α is λ-cofinal, $\lambda > 2^\omega$, and $|X| = 2^\omega$, there is a very good $\beta < \alpha$ such that $X \cap \Gamma_g^\alpha \subset \Gamma_g^\beta$. Let $\xi < \eta$ be such that $\mathcal{B}_\xi \subseteq \Gamma_f^\alpha\backslash\Gamma_f^\beta$ and $\xi \notin J_f^\beta$. It is not difficult to see that $\Pi(\mathcal{B}_\xi) \downarrow_{\Pi(\mathcal{A})} D$, and since $\Pi(\mathcal{C}_\eta) \subseteq D$, $\Pi(\mathcal{B}_\xi) \downarrow_{\Pi(\mathcal{C}_\eta)} D$.

Claim 4.3.1. *There is $a \in I_{\xi\eta}\backslash(I_{\xi\eta} \restriction \omega)$ such that $\Pi(a) \notin E$ and $\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi\cup\mathcal{C}_\eta)} E$.*

Proof of Claim 4.3.1. Suppose, towards a contradiction, that for every $a \in I_{\xi\eta}\backslash(I_{\xi\eta} \restriction \omega)$, $\Pi(a) \not\downarrow_{\Pi(\mathcal{B}_\xi\cup\mathcal{C}_\eta)} E$. Then, for every $a \in I_{\xi\eta}\backslash(I_{\xi\eta} \restriction \omega)$ there is $b_a \in E$ such that $\Pi(a) \not\downarrow_{\Pi(\mathcal{B}_\xi\cup\mathcal{C}_\eta)} b_a$. The model E was defined as an a-primary model over $\Gamma_g^{\alpha+1} \cup X$, therefore $|E| \leq \lambda(T) + (|\Gamma_g^{\alpha+1} \cup X| + \omega)^{<\omega}$. Since $\lambda(T) \leq 2^\omega$ and $|X| = 2^\omega$, we obtain $|E| \leq 2^\omega + |\Gamma_g^{\alpha+1}|$, by Fact 3.6, we get $|E| \leq g(\alpha)$ and $|E| < f(\alpha)$. Since $|I_{\xi\eta}| = f(\alpha)$, then there is $b \in E$ and $J = \{c_i \mid i < \omega\}$, a subset of $I_{\xi\eta}\backslash(I_{\xi\eta} \restriction \omega)$ such that for every $i < \omega$, $\Pi(c_i) \not\downarrow_{\Pi(\mathcal{B}_\xi\cup\mathcal{C}_\eta)} b$ holds. Since $\Pi(I_{\xi\eta}\backslash(I_{\xi\eta} \restriction \omega))$ is independent over $\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)$, then $b \not\downarrow_{\Pi(\mathcal{B}_\xi\cup\mathcal{C}_\eta)\cup\{\Pi(c_j) \mid j<i\}} \Pi(c_i)$ for every $i < \omega$. So T is not superstable, a contradiction. This finishes the proof of Claim 4.3.1.

Notice that $\Pi(I_{\xi\eta})$ is indiscernible over $\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)$. Since $\Pi(\mathcal{B}_\xi) \downarrow_{\Pi(\mathcal{C}_\eta)} D$, then by domination we get $M_3 \downarrow_{\Pi(\mathcal{C}_\eta)} D$, where M_3 is an a-primary model over $\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)$. So the models $M_0 = M_0' = \Pi(\mathcal{A})$, $M_1 = M_1' = \Pi(\mathcal{B}_\xi)$, $M_2 = \Pi(\mathcal{C}_\eta)$ and $M_2' = D$ satisfy the assumptions of Lemma 2.11, therefore $\Pi(I_{\xi\eta})$ is indiscernible over $\Pi(\mathcal{B}_\xi) \cup D$. By Remark 2.12, if M_3' is an a-primary model over $\Pi(\mathcal{B}_\xi) \cup D$ with $\Pi(I_{\xi\eta} \restriction \omega) \subseteq M_3'$, then $Av(\Pi(I_{\xi\eta} \restriction \omega), M_3') \perp D$ and $\Pi(I_{\xi\eta})$ is independent over $\Pi(\mathcal{B}_\xi) \cup D$. So, if a is the element given in Claim 4.3.1 and $\Pi(a) \notin M_3'$ holds, then $tp(\Pi(a), M_3') \perp D$.

Claim 4.3.2. $tp(\Pi(a), E) \perp D$

Proof of Claim 4.3.2. Let M_3' be an a-primary model over $\Pi(\mathcal{B}_\xi) \cup D$ with $\Pi(I_{\xi\eta} \restriction \omega) \subseteq M_3'$. Since E is a-saturated, then there is $\mathcal{F} M_3' \to E$ an elementary embedding such that $\mathcal{F} \restriction \Pi(\mathcal{B}_\xi) \cup D = id$. Let b be such that $b \models \mathcal{F}(Av(\Pi(I_{\xi\eta} \restriction \omega), M_3'))$, since $Av(\Pi(I_{\xi\eta} \restriction \omega), M_3') \perp D$, then $tp(b, \mathcal{F}(M_3')) \perp D$. By the way $I_{\xi\eta}$ was chosen and Remark 2.12, we know that $\Pi(I_{\xi\eta})$ is independent over $\Pi(\mathcal{B}_\xi) \cup D$, by Lemma 2.9 we conclude that $\mathcal{F}(Av(\Pi(I_{\xi\eta} \restriction \omega), M_3'))$ doesn't fork over $\Pi(\mathcal{B}_\xi) \cup D$. On the other hand, by Claim 4.3.1 $\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi\cup\mathcal{C}_\eta)} E$, so $\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi)\cup D} \mathcal{F}(M_3')$. By Fact 2.7, since $tp(b, \mathcal{F}(M_3')) \perp D$, $b \downarrow_{\Pi(\mathcal{B}_\xi)\cup D} \mathcal{F}(M_3')$ and $\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi)\cup D} \mathcal{F}(M_3')$ hold, then $tp(\Pi(a), \mathcal{F}(M_3')) \perp D$.

To show that $tp(\Pi(a), E) \perp D$ let d and B be such that $d \downarrow_D E$, $D \subseteq B$, $\Pi(a) \downarrow_E B$, and $d \downarrow_E B$. By transitivity, $d \downarrow_D E$ and $d \downarrow_E B$ implies that $d \downarrow_D E \cup B$. By Claim 4.3.1 we know that $\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)} E$, then by transitivity we get $\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)} E \cup B$. Therefore $d \downarrow_D \mathcal{F}(M_3') \cup B$ and $\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi) \cup D} \mathcal{F}(M_3') \cup B$ hold, so $d \downarrow_D \mathcal{F}(M_3')$, $d \downarrow_{\mathcal{F}(M_3')} B$ and $\Pi(a) \downarrow_{\mathcal{F}(M_3')} B$ hold. Since $tp(\Pi(a), \mathcal{F}(M_3')) \perp D$, we conclude that $\Pi(a) \downarrow_B b$, finishing the proof of Claim 4.3.2.

Let I_X be the set $\bigcup\{\mathcal{B}_r, I_{rp} \mid \mathcal{B}_r \not\subseteq \Gamma_g^{\alpha+1} \wedge r < p \wedge \mathcal{C}_p \subseteq X \backslash \Gamma_g^{\alpha+1}\}$. Let us show that $D \downarrow_X I_X \cup \Gamma_g^{\alpha+1}$. If $D \not\downarrow_X I_X \cup \Gamma_g^{\alpha+1}$, then there are finite $c \in D$ and $b \in (I_X \cup \Gamma_g^{\alpha}) \backslash X$ such that $a \not\downarrow_X b$.

Since D is a-constructable over X, then it is a-atomic over X. So, there is a finite $A_1 \subseteq X$ such that $stp(c, A_1) \vdash tp(c, X)$. Since T is superstable, there is a finite $A_2 \subseteq X$ such that $c \cup b \downarrow_{A_2} X$. Denote by A the set $A_1 \cup A_2$. Since X is nice, A is a finite subset of X and $b \in (I_X \cup \Gamma_g^{\alpha}) \backslash X$, then by Lemma 3.9 there is $\mathcal{F} \in Saut(\mathcal{M}, A)$ such that $\mathcal{F}(b) \in X$. Therefore $stp(\mathcal{F}(c), A_1) \vdash tp(c, X)$, and $\mathcal{F}(c) \downarrow_{A_1} X$, we conclude $\mathcal{F}(c) \downarrow_A \mathcal{F}(b)$ and $c \downarrow_A b$. Since $c \cup b \downarrow_{A_2} X$, then $c \cup b \downarrow_A X$. Therefore $c \downarrow_X b$, a contradiction.

By Fact 3.11, we know that $I_X \cup X \downarrow_{X \cap \Gamma_g^{\alpha+1}} \Gamma_g^{\alpha+1}$, then $I_X \downarrow_X \Gamma_g^{\alpha+1}$. Since $D \downarrow_X I_X \cup \Gamma_g^{\alpha+1}$, we conclude that $I_X \downarrow_D \Gamma_g^{\alpha+1}$. By the way E was chosen, we know that E is a-constructible over $D \cup \Gamma_g^{\alpha+1}$. Since D is a-saturated, we get that $\Gamma_g^{\alpha+1} \rhd_D E$. By domination we conclude $I_X \downarrow_D E$. Therefore, for every $c \in I_X$ we have that $c \downarrow_D E$. Since $c \downarrow_E E$ and $\Pi(a) \downarrow_E E$ hold, then by Claim 4.3.2 we conclude that $c \downarrow_E \Pi(a)$ for every $c \in I_X$. By the finite character we get $I_X \downarrow_E \Pi(a)$. By the way F was chosen, we know that F is a-constructible over $I_X \cup E$, and since E is a-saturated, we conclude that $I_X \rhd_E F$. Therefore $F \downarrow_E \Pi(a)$. Since $\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)} E$, by transitivity we conclude $\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)} F$.

On the other hand $\Pi(a) \in \mathcal{A}^g$ and $\mathcal{A}^g$ is a-constructable over $F \cup \Gamma_g$, then $\mathcal{A}^g$ is a-atomic over $F \cup \Gamma_g$ and there is a finite $B \subseteq F \cup \Gamma_g$ such that $(tp(\Pi(a), F \cup \Gamma_g), B) \in F_\omega^a$ and $\Pi(a) \in \mathcal{N}$, where $\mathcal{N} \subseteq \mathcal{A}^g$ is a-primary over $F \cup B$. Let $B' = B \backslash F$, there is a nice set $\mathcal{Y}$ such that $\mathcal{Y} \cap F = \mathcal{A}$, $B' \subseteq \mathcal{Y}$, $\mathcal{Y}$ Γ_g-nice for all (r,p) that satisfy $\mathcal{B}_r, \mathcal{C}_p \subset \mathcal{Y}$, and $S = \{r \in J_g \mid (r \in (J_g)_{<\lambda} \wedge \mathcal{B}_r \subset \mathcal{Y}) \vee (r \in (J_g)_\lambda \wedge \mathcal{C}_r \subset \mathcal{Y})\}$ is finite. Define $\mathcal{X} = \{r \in J_g \mid (r \in (J_g)_{<\lambda} \wedge \mathcal{B}_r \subset X) \vee (r \in (J_g)_\lambda \wedge \mathcal{C}_r \subset X)\}$. Let $\bar{S} = S \cup \{r \in (J_g)_{<\lambda} \mid \exists p \in S\ (r < p)\}$ and $\bar{\mathcal{X}} = \mathcal{X} \cup \{r \in (J_g)_{<\lambda} \mid \exists p \in \mathcal{X}\ (r < p)\}$. By the way $\bar{\mathcal{X}}$ was defined, we know that for every limit ordinal $\theta < \lambda$ and $\zeta \in J_g$, if for all $\theta' < \theta$, $\zeta \restriction \theta' \in \bar{\mathcal{X}}$ holds, then $\zeta \restriction \theta \in \bar{\mathcal{X}}$. Notice that since $cf(\alpha) = \lambda$, if $\theta < \lambda$ is a limit ordinal such that for all $\theta' < \theta$, $\zeta \restriction \theta' \in J_g^{\alpha+1}$ holds, then $\zeta \restriction \theta \in J_g^{\alpha+1}$. We conclude that if $\theta < \lambda$ and $\zeta \in J_g$ are such that for all $\theta' < \theta$, $\zeta \restriction \theta' \in \bar{\mathcal{X}} \cup J_g^{\alpha+1}$ and $\zeta \restriction \theta \in \bar{S} \backslash (\bar{\mathcal{X}} \cup J_g^{\alpha+1})$, then θ is a successor ordinal. Let $\{u_i\}_{i<f(\alpha)^+}$ be a sequence of subtrees of J_g with the following properties:

- $u_0 = \bar{S}$
- Every u_i is a tree isomorphic to u_0.
- If $i \neq j$, then $u_i \cap u_j = u_0 \cap (\bar{\mathcal{X}} \cup J_g^{\alpha+1})$.
- Every $\zeta \in dom(c_g) \cap u_0$ satisfies $c_f(\zeta) = c_f(G_i(\zeta))$, where G_i is the isomorphism between u_0 and u_i.

For every $\zeta \in u_0$ and $\theta < \lambda$ such that $\zeta \restriction \theta \in \bar{\mathcal{X}} \cup J_g^{\alpha+1}$ and $\zeta \restriction \theta + 1 \in u_0 \backslash (\bar{\mathcal{X}} \cup J_g^{\alpha+1})$, it holds by Definition 2.2 that $\zeta \restriction \theta$ has κ many immediate successors in $J_g \backslash J_g^{\alpha+1}$. Also by Definition 2.2 the elements of J_f are all the functions $\eta : s \to \lambda \times \kappa^4$ that satisfy the items 1 to 8, therefore each of the immediate successors of $\zeta \restriction \gamma$, ζ', satisfies that in the set $\{r \in J_f \mid \zeta' \leq r\}$ there is a subtree isomorphic (as coloured tree) to $\{p \in u_0 \backslash (\bar{\mathcal{X}} \cup J_g^{\alpha+1}) \mid \zeta \restriction \gamma + 1 \leq p\}$. This and the fact that S is finite, gives the existence of the sequence $\{u_i\}_{i<f(\alpha)^+}$. By the way we chose the sequence $\{u_i\}_{i<f(\alpha)^+}$, for every $i < f(\alpha)^+$, the isomorphism G_i induces a coloured trees isomorphism $\bar{G}_i : \bar{\mathcal{X}} \cup J_g^{\alpha+1} \cup u_0 \to \bar{\mathcal{X}} \cup J_g^{\alpha+1} \cup u_i$ such that $\bar{G}_i \restriction \bar{\mathcal{X}} \cup J_g^{\alpha+1} = id$. Let us denote by z_i the tree $\bar{\mathcal{X}} \cup J_g^{\alpha+1} \cup u_i$.

Let us define $U_i = \{\mathcal{B}_r \mid r \in z_i \wedge r \in (J_g)_{<\lambda}\} \cup \{\mathcal{C}_p \mid p \in z_i \wedge p \in (J_g)_\lambda\}$ and $\bar{U}_i = U_i \cup \{I_{rp} \mid \mathcal{B}_r \in U_i \wedge \mathcal{C}_p \in U_i \wedge r < p\}$. Notice that $\bigcup \bar{U}_i$ is nice for all $i < f(\alpha)^+$. Since u_i is isomorphic to $\bar{S}$, then $p \in z_i$ and $r < p$, implies $r \in z_i$. Therefore, $\bigcup \bigcup_{j \neq i} \bar{U}_j$ is nice for all $i < f(\alpha)^+$.

Claim 4.3.3. *For all $i < f(\alpha)^+$ it holds that $\bigcup \bar{U}_i \downarrow_F \bigcup \bigcup_{j \neq i} \bar{U}_j$.*

Proof of Claim 4.3.3. By the way the sets $\bar{U}_i$ were constructed, we know that $(\bigcup \bar{U}_i) \cap (\bigcup \bar{U}_j) = \Gamma_g^{\alpha+1} \cup X \cup I_X$ for all $i \neq j$. Let us denote by $\mathbb{F}$ the set $\Gamma_g^{\alpha+1} \cup X \cup I_X$. By Corollary 4.13 we know that

$$\bigcup \bar{U}_i \downarrow_{\mathbb{F}} \bigcup \bigcup_{j \neq i} \bar{U}_j.$$

Let us proof that $F \downarrow_{\mathbb{F}} \bigcup \bigcup_{j < f(\alpha)^+} \bar{U}_j$. Suppose it is false, then $F \not\downarrow_{\mathbb{F}} \bigcup \bigcup_{j < f(\alpha)^+} \bar{U}_j$ and there are finite $c \in F$ and $b \in \bigcup \bigcup_{j < f(\alpha)^+} \bar{U}_j$ such that $c \not\downarrow_{\mathbb{F}} b$. Since F is a-constructable over $\mathbb{F}$, then it is a-atomic over $\mathbb{F}$. So, there is a finite $A_1 \subseteq \mathbb{F}$ such that $stp(c, A_1) \vdash tp(c, \mathbb{F})$. Since T is superstable, there is a finite $A_2 \subseteq \mathbb{F}$ such that $c \cup b \downarrow_{A_2} \mathbb{F}$. Denote by A the set $A_1 \cup A_2$. By Lemma 3.9 there is $\mathcal{F} \in Saut(\mathcal{M}, A)$ such that $\mathcal{F}(b) \in \mathbb{F}$. Therefore $stp(\mathcal{F}(c), A_1) \vdash tp(c, \mathbb{F})$, and $\mathcal{F}(c) \downarrow_{A_1} \mathbb{F}$. So $\mathcal{F}(c) \downarrow_A \mathcal{F}(b)$ and $c \downarrow_A b$. Since $c \cup b \downarrow_{A_2} \mathbb{F}$, then $c \cup b \downarrow_A \mathbb{F}$. Therefore $c \downarrow_{\mathbb{F}} b$, a contradiction.

Since $F \downarrow_{\mathbb{F}} \bigcup \bigcup_{j < f(\alpha)^+} \bar{U}_j$ and $\bigcup \bar{U}_i \downarrow_{\mathbb{F}} \bigcup \bigcup_{j \neq i} \bar{U}_j$ holds, we conclude that $\bigcup \bar{U}_i \downarrow_F \bigcup \bigcup_{j \neq i} \bar{U}_j$, finishing the proof of Claim 4.3.3.

The isomorphisms $(\bar{G}_i)_{i < f(\alpha)^+}$ induce the following elementary maps $\mathcal{H}_{rp}^i : \mathcal{B}_r \cup \mathcal{C}_p \cup I_{rp} \to \mathcal{B}_{\bar{G}_i(r)} \cup \mathcal{C}_{\bar{G}_i(p)} \cup I_{\bar{G}_i(r)\bar{G}_i(p)}$ for all $r, p \in z_0$ ($r \in (J_g)_{<\lambda}$ and $p \in (J_g)_\lambda$), given by $\mathcal{H}_{rp}^i = H_{\bar{G}_i(r)\bar{G}_i(p)} \circ H_{rp}^{-1}$. Let $\{D_i \mid i < \theta\}$ be an enumeration of U_0 such that if D_i is a subset of $\Gamma_g^{\alpha+1} \cup X \cup I_X$ and D_j is a subset of $U_0 \backslash \Gamma_g^{\alpha+1} \cup X \cup I_X$, then $i < j$. Let $\{D'_i \mid i < \theta'\}$ be an enumeration of $\{I_{rp} \mid I_{rp} \in \bar{U}_0\}$.

It is easy to check that the map $\mathcal{H}_i : \bigcup \bar{U}_0 \to \bigcup \bar{U}_i$ defined by

$$\mathcal{H}_i = \bigcup_{\eta \in z_0 \cap (J_f)_\lambda} \bigcup_{\xi \in z_0 \cap (J_f)_{<\lambda}, \xi < \eta} \mathcal{H}_{\xi\eta}^i$$

is elementary. Notice that for any permutation $\mathcal{R} : f(\alpha)^+ \to f(\alpha)^+$ and any $i < f(\alpha)^+$, $tp(\bigcup \bigcup_{j<i} \bar{U}_j, \Gamma_g^{\alpha+1} \cup X \cup I_X) = tp(\bigcup \bigcup_{j<i} \bar{U}_{\mathcal{R}(j)}, \Gamma_g^{\alpha+1} \cup X \cup I_X)$ holds.

Therefore $(\bigcup \bar{U}_i)_{i < f(\alpha)^+}$ is an indiscernible sequence over $\Gamma_g^{\alpha+1} \cup X \cup I_X$. So, for all $i < f(\alpha)^+$, $stp(\bigcup \bar{U}_0, \Gamma_g^{\alpha+1} \cup X \cup I_X) = stp(\bigcup \bar{U}_i, \Gamma_g^{\alpha+1} \cup X \cup I_X)$. Let $\mathcal{G}_i : F \cup \bigcup \bar{U}_0 \to F \cup \bigcup \bar{U}_i$, be given by $\mathcal{G}_i \restriction F = id$ and $\mathcal{G}_i \restriction \bigcup \bar{U}_0 = \mathcal{H}_i$. It is easy to check that $\mathcal{G}_i$ is elementary.

Let us define for all $i < f(\alpha)^+$ the model $M_i \subseteq \mathcal{A}^g$ as an a-primary model over $F \cup \bigcup_{j<i} M_j \cup \bigcup \bar{U}_i$, with $\mathcal{N} \subseteq M_0$ and let $b_0 \in M_0$ be $\Pi(a)$ (notice that $B \subseteq \bar{U}_0$ was chosen such that $(tp(\Pi(a), F \cup \Gamma_g), B) \in F_\omega^a$ and $\Pi(a) \in \mathcal{N}$, $\mathcal{N}$ is the a-primary model over $F \cup B$). For all $0 < i < f(\alpha)^+$ let $\bar{\mathcal{G}}_i \in Saut(\mathcal{M}, \Gamma_g^{\alpha+1} \cup X \cup I_X)$ be such that $\bar{\mathcal{G}}_i \restriction F \cup \bigcup \bar{U}_i = \mathcal{G}_i \restriction F \cup \bigcup \bar{U}_i$ and $b_i \in M_i$ be such that $stp(b_i, \mathcal{G}_i(B)) = stp(\bar{\mathcal{G}}_i(\Pi(a)), \mathcal{G}_i(B))$. We know that $(tp(\Pi(a), F \cup \Gamma_g), B) \in F_\omega^a$, so by a-isolation and the definition of $\bar{\mathcal{G}}_i$ we conclude that $(tp(b_i, \bar{\mathcal{G}}_i(F \cup \bigcup \bar{U}_0)), \mathcal{G}_i(B)) \in F_\omega^a$, so $(tp(b_i, F \cup \bigcup \bar{U}_i), \mathcal{G}_i(B)) \in F_\omega^a$. Therefore $tp(b_i, F) = tp(\bar{\mathcal{G}}_i(\Pi(a)), F)$ and since $\bar{\mathcal{G}}_i$ is an automorphism that fix F, we conclude that $tp(b_i, F) = tp(\Pi(a), F)$. On the other hand $(tp(b_i, F \cup \bigcup \bar{U}_i), \mathcal{G}_i(B)) \in F_\omega^a$ implies that $b_i \cup F \cup \bigcup \bar{U}_i$ is a-constructable over $F \cup \bigcup \bar{U}_i$, since F is a-saturated then $\bigcup \bar{U}_i \triangleright_F b_i \cup \bigcup \bar{U}_i$. By Claim 4.3.3 we know that

$\bigcup \bar{U}_i \downarrow_F \bigcup\bigcup_{j\neq i} \bar{U}_j$, so by domination we conclude that $b_i \cup \bigcup \bar{U}_i \downarrow_F \bigcup\bigcup_{j\neq i} \bar{U}_j$, in particular $b_i \downarrow_F \bigcup\bigcup_{j\neq i} \bar{U}_j$ holds for all $i < f(\alpha)^+$.

Notice that for all $i < f(\alpha)^+$, M_i is a-constructable over $F \cup \bigcup\bigcup_{j\leq i} \bar{U}_j$. Therefore $\bigcup\bigcup_{k\leq j} \bar{U}_k \rhd_F M_j$ holds for all $i < f(\alpha)^+$, and since $b_i \downarrow_F \bigcup\bigcup_{j\neq i} \bar{U}_j$ holds for all $i < f(\alpha)^+$, then $b_i \downarrow_F M_j$ holds for all $j, i < f(\alpha)^+$, $j < i$. In particular $b_i \downarrow_F \bigcup_{k\leq j} b_k$ holds for all $j, i < f(\alpha)^+$, $j < i$. We conclude that $b_i \downarrow_F \bigcup_{j<i} b_j$ holds for all $i < f(\alpha)^+$. Since $tp(b_i, F) = tp(\Pi(a), F)$ and $\Pi(a) \downarrow_{\Pi(\mathcal{B}_\xi\cup\mathcal{C}_\eta)} F$, we get that $b_i \downarrow_{\Pi(\mathcal{B}_\xi\cup\mathcal{C}_\eta)} F$ and by transitivity we conclude that $b_i \downarrow_{\Pi(\mathcal{B}_\xi\cup\mathcal{C}_\eta)} \bigcup_{j<i} b_j$. So $(b_i)_{i<f(\alpha)^+}$ is an independent sequence over $\Pi(\mathcal{B}_\xi \cup \mathcal{C}_\eta)$. Since for $i \neq j$ we know that $tp(b_i, F) = tp(b_j, F)$, the types over F are stationary, and $b_i \downarrow_F \bigcup_{j<i} b_j$, then we conclude that $(b_i)_{i<f(\alpha)^+}$ is an indiscernible sequence over F.

For every $i < f(\alpha)^+$ let c_i be $\Pi^{-1}(b_i)$, since Π is an isomorphism, then $(c_i)_{i<f(\alpha)^+}$ is an indiscernible sequence over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$ and an independent sequence over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$, notice that $c_0 = a$, so $c_0 \in I_{\xi\eta}$.

Denote by J the sequence $(c_i)_{i<f(\alpha)^+}$, since T is superstable, there is $J' \subseteq J$ of power $f(\alpha)^+$ such that $c_0 \notin J'$ and satisfies $J' \downarrow_{J\restriction\omega\cup\mathcal{B}_\xi\cup\mathcal{C}_\eta} I_{\xi\eta}$. Since J is an independent sequence over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$, then $J' \downarrow_{\mathcal{B}_\xi\cup\mathcal{C}_\eta} J \restriction \omega \cup I_{\xi\eta}$. Let us denote by Q the set $\mathcal{B}_\xi \cup \mathcal{C}_\eta \cup (I_{\xi\eta} \restriction \omega)\backslash\{c_0\}$, so $J' \downarrow_Q I_{\xi\eta}$. Since $Av(I_{\xi\eta}, Q)$ is stationary and $I_{\xi\eta}$ is independent over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$, we conclude that $I' = \{c_0\} \cup (I_{\xi\eta}\backslash(I_{\xi\eta} \restriction \omega))$ is indiscernible over $J' \cup Q$. Especially I' is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta \cup J'$. On the other hand $J' \downarrow_{\mathcal{B}_\xi\cup\mathcal{C}_\eta} J \restriction \omega \cup I_{\xi\eta}$ implies that $J' \downarrow_{\mathcal{B}_\xi\cup\mathcal{C}_\eta} I_{\xi\eta}$, and since $I_{\xi\eta}$ is independent over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$, we conclude that $I_{\xi\eta}$ is independent over $\mathcal{B}_\xi \cup \mathcal{C}_\eta \cup J'$. In particular I' is independent over $\mathcal{B}_\xi \cup \mathcal{C}_\eta \cup J'$. We will prove by induction that $J' \cup I'$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$. Let us denote by $\{d_i \mid i < f(\alpha)\}$ the sequence I'. Since $c_0 \in I' \cap J$, $c_0 \models Av(J', \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup J')$, and I' is indiscernible over $J' \cup Q$, then for every $i < f(\alpha)$,

$$d_i \models Av(J', \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup J').$$

Suppose j is such that for all $n < j$ the sequence $J' \cup \{d_i \mid i \leq n\}$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$, then $J' \cup \{d_i \mid i < j\}$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$, therefore $Av(J' \cup \{d_i \mid i < j\}, \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup J' \cup \{d_i \mid i < j\}) = Av(J', \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup J' \cup \{d_i \mid i < j\})$ and it does not fork over $\mathcal{B}_\xi \cup \mathcal{C}_\eta \cup J'$. On the other hand we know that $Av(J', \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup J')$ is stationary, $d_j \downarrow_{\mathcal{B}_\xi\cup\mathcal{C}_\eta\cup J'} \{d_i \mid i < j\}$ and $d_j \models Av(J', \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup J')$, we conclude that $tp(d_j, \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup J' \cup \{d_i \mid i < j\})) = Av(J' \cup \{d_i \mid i < j\}, \mathcal{B}_\xi \cup \mathcal{C}_\eta \cup J' \cup \{d_i \mid i < j\})$. Therefore $J' \cup \{d_i \mid i \leq j\}$ is indiscernible over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$. We conclude that $J' \cup I'$ is indiscernible. So J' is equivalent to $I_{\xi\eta}$ and for all $d \in J'$, $d \models Av(I_{\xi\eta} \restriction \omega, I_{\xi\eta} \restriction \omega \cup \mathcal{B}_\xi \cup \mathcal{C}_\eta)$. Since J' is independent over $\mathcal{B}_\xi \cup \mathcal{C}_\eta$ and $J' \downarrow_{\mathcal{B}_\xi\cup\mathcal{C}_\eta} I_{\xi\eta}$, we conclude that J' is independent over $I_{\xi\eta} \restriction \omega \cup \mathcal{B}_\xi \cup \mathcal{C}_\eta$, thus $dim(p_{\xi\eta}, \mathcal{A}^f) \geq f(\alpha)^+$, but this contradicts Lemma 3.7. □

5 Corollaries

Corollary 5.1. *If κ is innaccessible, and T is a theory with S-DOP, then $E^\kappa_{\lambda\text{-club}} \leq_c \cong_T$.*

Proof. Let f and g be elements of κ^κ. First we will construct a function $F : \kappa^\kappa \to \kappa^\kappa$ such that $f\ E^\kappa_{\lambda\text{-club}}\ g$ if and only if $\mathcal{A}^{F(f)}$ and $\mathcal{A}^{F(g)}$ are isomorphic.

For every cardinal $\alpha < \kappa$, define $S_\alpha = \{\beta \in Card \cap \kappa \mid \lambda, \alpha^{+++}, \alpha^\lambda < \beta\}$. Let $\mathcal{G}_\beta$ be a bijection from κ into S_β, for every $\beta < \kappa$. For every $f \in \kappa^\kappa$ define $F(f)$ by $F(f)(\beta) = \mathcal{G}_\beta(f(\beta))$, for every $\beta < \kappa$. Clearly $f\ E^\kappa_{\lambda\text{-club}}\ g$ if and only if $F(f)\ E^\kappa_{\lambda\text{-club}}\ F(g)$ i.e. $\mathcal{A}^{F(f)}$ and $\mathcal{A}^{F(g)}$ are isomorphic and F is continuous.

Finally we need to find $\mathcal{G} : \{F(f) \mid f \in \kappa^\kappa\} \to \kappa^\kappa$ such that $\mathcal{A}_{\mathcal{G}(F(f))} \cong \mathcal{A}^{F(f)}$ and $f \mapsto \mathcal{G}(F(f))$ is continuous. Notice that for every $f, g \in \kappa^\kappa$ and $\alpha < \kappa$, by Definition 2.2 and the definition of J^α_f in

Remark 2.4, it holds:

$$F(f)\restriction\alpha = F(g)\restriction\alpha \Leftrightarrow J^\alpha_{F(f)} = J^\alpha_{F(g)}.$$

By Definition 3.5, for every $f, g \in \kappa^\kappa$ and $\alpha < \kappa$ it holds:

$$J^\alpha_{F(f)} = J^\alpha_{F(g)} \Leftrightarrow \Gamma^\alpha_{F(f)} = \Gamma^\alpha_{F(g)}.$$

By the definition of $\mathcal{A}^\alpha_f$ in Theorem 4.1, for every $f, g \in \kappa^\kappa$ and $\alpha < \kappa$ an $F(f)$-good and $F(g)$-good cardinal, it holds:

$$\Gamma^\alpha_{F(f)} = \Gamma^\alpha_{F(g)} \Leftrightarrow \mathcal{A}^\alpha_{F(f)} \cong \mathcal{A}^\alpha_{F(g)}.$$

In general,since there are club many $F(f)$-good and $F(g)$-good cardinals, then by the definition of $\mathcal{A}^\alpha_f$ in Theorem 4.1 we can construct the models $\mathcal{A}^f$ such that for every $f, g \in \kappa^\kappa$ and $\alpha < \kappa$, it holds:

$$J^\alpha_{F(f)} = J^\alpha_{F(g)} \Leftrightarrow \mathcal{A}^\alpha_{F(f)} = \mathcal{A}^\alpha_{F(g)}.$$

So we can construct the models $\mathcal{A}^f$ such that for every $f, g \in \kappa^\kappa$ and $\alpha < \kappa$, it holds:

$$F(f)\restriction\alpha = F(g)\restriction\alpha \Leftrightarrow \mathcal{A}^\alpha_{F(f)} = \mathcal{A}^\alpha_{F(g)}.$$

For every $f \in \kappa^\kappa$ define $C_f \subseteq Card \cap \kappa$ such that $\forall\alpha \in C_f$, it holds that for all β ordinal smaller than α, $\mid \mathcal{A}^\beta_{F(f)} \mid<\mid \mathcal{A}^\alpha_{F(f)} \mid$. For every $f \in \kappa^\kappa$ and $\alpha \in C_f$ choose $E^\alpha_f : dom(\mathcal{A}^\alpha_{F(f)}) \to\mid \mathcal{A}^\alpha_{F(f)} \mid$ a bijection, such that $\forall\beta, \alpha \in C_f$, $\beta < \alpha$ it holds that $E^\beta_f \subseteq E^\alpha_f$. Therefore $\bigcup_{\alpha\in C_f} E^\alpha_f = E_f$ is such that $E_f : dom(\mathcal{A}^{F(f)}) \to \kappa$ is a bijection, and for every $f, g \in \kappa^\kappa$ and $\alpha < \kappa$ it holds: If $F(f)\restriction\alpha = F(g)\restriction\alpha$, then $E_f \restriction dom(\mathcal{A}^\alpha_{F(f)}) = E_g \restriction dom(\mathcal{A}^\alpha_{F(g)})$.

Let π be the bijection in Definition 1.6, define the function $\mathcal{G}$ by:

$$\mathcal{G}(F(f))(\alpha) = \begin{cases} 1 & \text{if } \alpha = \pi(m, a_1, a_2, \ldots, a_n) \text{ and } \mathcal{A}^{F(f)} \models P_m(E^{-1}_f(a_1), E^{-1}_f(a_2), \ldots, E^{-1}_f(a_n)) \\ 0 & \text{in other case.} \end{cases}$$

To show that $\mathcal{G}$ is continuous, let $[\eta\restriction\alpha]$ be a basic open set and $\xi \in \mathcal{G}^{-1}[[\eta\restriction\alpha]]$. So, there is $\beta \in C_\xi$ such that for all $\gamma < \alpha$, if $\gamma = \pi(m, a_1, a_2, \ldots, a_n)$, then $E^{-1}_\xi(a_i) \in dom(\mathcal{A}^\beta_\xi)$ holds for all $i \le n$. Since for all $\zeta \in [\xi\restriction\beta]$ it holds that $\mathcal{A}^\beta_\xi = \mathcal{A}^\beta_\zeta$, then for every $\gamma < \alpha$ that satisfies $\gamma = \pi(m, a_1, a_2, \ldots, a_n)$, it holds that:

$$\mathcal{A}^\xi \models P_m(E^{-1}_\xi(a_1), E^{-1}_\xi(a_2), \ldots, E^{-1}_\xi(a_n)) \Leftrightarrow \mathcal{A}^\zeta \models P_m(E^{-1}_\zeta(a_1), E^{-1}_\zeta(a_2), \ldots, E^{-1}_\zeta(a_n)).$$

We conclude that $\mathcal{G}(\zeta) \in [\eta\restriction\alpha]$, and $\mathcal{G}$ is continuous. □

In [HM] it was proved that if T is a classifiable theory and $\mu < \kappa$ is a regular cardinal, then $\cong_T$ is continuously reducible to $E^\kappa_{\mu\text{-}club}$.

Corollary 5.2. *If κ is an innaccessible and T_1 is a classifiable theory and T_2 is a superstable theory with S-DOP, then $\cong_{T_1} \le_c \cong_{T_2}$.*

The last corollaries are about Σ^1_1-completeness. Suppose E is an equivalence relation on κ^κ. We say that E is Σ^1_1 if E is the projection of a closed set in $\kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa$ and it is Σ^1_1-complete, if every Σ^1_1 equivalence relation is Borel reducible to E.

In [HK] it was proved, under the assumption $V = L$, that $E^\kappa_{\mu\text{-club}}$ is Σ^1_1-complete for all regular $\mu < \kappa$. In [FMR], under the assumption GCH, it was proved that there exists a cofinality-preserving GCH-preserving forcing extension in which $E^\kappa_{\mu\text{-club}}$ is Σ^1_1-complete for all regular $\mu < \kappa$.

Corollary 5.3.
- *Suppose $V = L$. If κ is an innaccessible and T is a superstable theory with S-DOP, then $\cong_T$ is Σ^1_1-complete.*
- *Suppose GCH. There exists a cofinality-preserving GCH-preserving forcing extension in which If κ is an innaccessible and T is a superstable theory with S-DOP, then $\cong_T$ is Σ^1_1-complete.*

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