A Borel-reducibility Counterpart of Shelah's Main Gap Theorem

Tapani Hyttinen, Vadim Kulikov, Miguel Moreno
University of Helsinki

December 16, 2015

Abstract

We study the Borel-reducibility of isomorphism relations of complete first order theories and show the consistency of the following: For all such theories T and T′, if T is classifiable and T′ is not, then the isomorphism of models of T′ is strictly above the isomorphism of models of T with respect to Borel-reducibility. In fact, we can also ensure that a range of equivalence relations modulo various non-stationary ideals are strictly between those isomorphism relations. The isomorphism relations are considered on models of some fixed uncountable cardinality obeying certain restrictions.

1 Introduction

Throughout this article we assume that κ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$. The generalized Baire space is the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set. The open sets are of the form $\bigcup X$ where X is a collection of basic open sets. The collection of κ-Borel subsets of κ^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ. A κ-Borel set is any element of this collection. We usually omit the prefix "$\kappa-$". In [Vau74] Vought studied this topology in the case $\kappa = \omega_1$ assuming CH and proved the following:

Theorem. *A set $B \subset \omega_1^{\omega_1}$ is Borel and closed under permutations if and only if there is a sentence φ in $L_{\omega_1^+\omega_1}$ such that $B = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$.*

This result was generalized in [FHK14] to arbitrary κ that satisfies $\kappa^{<\kappa} = \kappa$. Mekler and Väänänen continued the study of this topology in [MV93].

We will work with the subspace 2^κ with the relative subspace topology. A function $f\colon 2^\kappa \to 2^\kappa$ is *Borel*, if for every open set $A \subseteq 2^\kappa$ the inverse image $f^{-1}[A]$ is a Borel subset of 2^κ. Let E_1 and E_2 be equivalence relations on 2^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f\colon 2^\kappa \to 2^\kappa$ that satisfies $(x,y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$. We call f a *reduction* of E_1 to E_2. This is denoted by $E_1 \leq_B E_2$ and if f is continuous, then we say that E_1 is *continuously reducible* to E_2 and this is denoted by $E_1 \leq_c E_2$.

The following is a standard way to code structures with domain κ with elements of 2^κ. To define it, fix a countable relational vocabulary $\mathcal{L} = \{P_n \mid n < \omega\}$.

Definition 1.1. Fix a bijection $\pi\colon \kappa^{<\omega} \to \kappa$. For every $\eta \in 2^\kappa$ define the $\mathcal{L}$-structure $\mathcal{A}_\eta$ with domain κ as follows: For every relation P_m with arity n, every tuple $(a_1, a_2, \dots, a_n)$ in κ^n satisfies

$$(a_1, a_2, \dots, a_n) \in P_m^{\mathcal{A}_\eta} \iff \eta(\pi(m, a_1, a_2, \dots, a_n)) = 1.$$

Note that for every $\mathcal{L}$-structure $\mathcal{A}$ there exists $\eta \in 2^\kappa$ with $\mathcal{A} = \mathcal{A}_\eta$. For club many $\alpha < \kappa$ we can also code the $\mathcal{L}$-structures with domain α:

Definition 1.2. Denote by C_π the club $\{\alpha < \kappa \mid \pi[\alpha^{<\omega}] \subseteq \alpha\}$. For every $\eta \in 2^\kappa$ and every $\alpha \in C_\pi$ define the structure $\mathcal{A}_{\eta\restriction\alpha}$ with domain α as follows: For every relation P_m with arity n, every tuple $(a_1, a_2, \dots, a_n)$ in α^n satisfies

$$(a_1, a_2, \dots, a_n) \in P_m^{\mathcal{A}_{\eta\restriction\alpha}} \iff \eta \restriction_\alpha (\pi(m, a_1, a_2, \dots, a_n)) = 1.$$

For every $\alpha \in C_\pi$ and every $X \subseteq \alpha$ we will denote the structure $\mathcal{A}_F$ by $\mathcal{A}_X$, where F is the characteristic function of X. We will work with two equivalence relations on 2^κ: the isomorphism relation and the equivalence modulo the non-stationary ideal.

Definition 1.3 (The isomorphism relation). Assume T is a complete first order theory in a countable vocabulary. We define $\cong^\kappa_T$ as the relation

$$\{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T)\}.$$

We will omit the superscript "κ" in $\cong^\kappa_T$ when it is clear from the context. For every first order theory T in a countable vocabulary there is an isomorphism relation associated with T, $\cong^\kappa_T$. For every stationary $X \subset \kappa$, we define an equivalence relation modulo the non-stationary ideal associated with X:

Definition 1.4. For every $X \subset \kappa$ stationary, we define E_X as the relation

$$E_X = \{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (\eta^{-1}[1] \triangle \xi^{-1}[1]) \cap X \text{ is not stationary}\}$$

where $\triangle$ denotes the symmetric difference.

For every regular cardinal $\mu < \kappa$ denote $\{\alpha < \kappa \mid cf(\alpha) = \mu\}$ by S^κ_μ. A set C is *μ-club* if it is ubounded and closed under μ-limits, i.e. if $S^\kappa_\mu \setminus C$ is non-stationary. Accordingly, we will denote the equivalence relation E_X for $X = S^\kappa_\mu$ by $E^2_{\mu\text{-club}}$. Note that $(f, g) \in E^2_{\mu\text{-club}}$ if and only if the set $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\}$ contains a μ-club.

2 Reduction to E_X

Classifiable theories (superstable with NOTOP and NDOP) have a close connection to the Ehrenfeucht-Fraïssé games (EF-games for short). We will use them to study the reducibility of the isomorphism relation of classifiable theories. The following definition is from [HM15, Def 2.3]:

Definition 2.1 (The Ehrenfeucht-Fraïssé game). Fix an enumeration $\{X_\gamma\}_{\gamma<\kappa}$ of the elements of $\mathcal{P}_\kappa(\kappa)$ and an enumeration $\{f_\gamma\}_{\gamma<\kappa}$ of all the functions with both the domain and range in $\mathcal{P}_\kappa(\kappa)$. For every $\alpha \leqslant \kappa$ the game $\text{EF}^\alpha_\omega(\mathcal{A} \restriction_\alpha, \mathcal{B} \restriction_\alpha)$ on the restrictions $\mathcal{A} \restriction \alpha$ and $\mathcal{B} \restriction \alpha$ of the structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ is defined as follows: In the n-th move, first **I** chooses an ordinal $\beta_n < \alpha$ such that $X_{\beta_n} \subset \alpha$ and $X_{\beta_{n-1}} \subseteq X_{\beta_n}$. Then **II** chooses an ordinal $\theta_n < \alpha$ such that $dom(f_{\theta_n}), \text{ran}(f_{\theta_n}) \subset \alpha$,

$X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap \text{ran}(f_{\theta_n})$ and $f_{\theta_{n-1}} \subseteq f_{\theta_n}$ (if $n = 0$ then $X_{\beta_{n-1}} = \emptyset$ and $f_{\theta_{n-1}} = \emptyset$). The game ends after ω moves. Player **II** wins if $\bigcup_{i<\omega} f_{\theta_i} \colon A \restriction_\alpha \to B \restriction_\alpha$ is a partial isomorphism. Otherwise player **I** wins. If $\alpha = \kappa$ then this is the same as the standard EF-game which is usually denoted by EF^κ_ω.

When a player P has a winning strategy in a game G, we denote it by $P \uparrow G$.

The following lemma is proved in [HM15, Lemma 2.4] and is used in the main result of this section which in turn is central to the main theorem of this paper.

Lemma 2.2. *If $\mathcal{A}$ and $\mathcal{B}$ are structures with domain κ, then*

- $\mathbf{II} \uparrow \text{EF}^\kappa_\omega(\mathcal{A}, \mathcal{B}) \iff \mathbf{II} \uparrow \text{EF}^\alpha_\omega(\mathcal{A} \restriction_\alpha, \mathcal{B} \restriction_\alpha)$ *for club-many α,*
- $\mathbf{I} \uparrow \text{EF}^\kappa_\omega(\mathcal{A}, \mathcal{B}) \iff \mathbf{I} \uparrow \text{EF}^\alpha_\omega(\mathcal{A} \restriction_\alpha, \mathcal{B} \restriction_\alpha)$ *for club-many α.*

Remark 1. In [HM15, Lemma 2.7] it was proved that there exists a club C_{EF} of α such that the relation defined by the game

$$\{(\mathcal{A}, \mathcal{B}) \mid \mathbf{II} \uparrow \text{EF}^\alpha_\omega(\mathcal{A} \restriction_\alpha, \mathcal{B} \restriction_\alpha)\}$$

is an equivalence relation.

Remark 2. Shelah proved in [She90], that if T is classifiable then every two models of T that are $L_{\infty,\kappa}$-equivalent are isomorphic. On the other hand $L_{\infty,\kappa}$-equivalence is equivalent to EF^κ_ω-equivalence. So for every two models $\mathcal{A}$ and $\mathcal{B}$ of T we have $\mathbf{II} \uparrow \text{EF}^\kappa_\omega(\mathcal{A}, \mathcal{B}) \iff \mathcal{A} \cong \mathcal{B}$ and $\mathbf{I} \uparrow \text{EF}^\kappa_\omega(\mathcal{A}, \mathcal{B}) \iff \mathcal{A} \not\cong \mathcal{B}$.

Lemma 2.3. *Assume T is a classifiable theory and $\mu < \kappa$ is a regular cardinal. If $\diamondsuit_\kappa(X)$ holds then $\cong^\kappa_T$ is continuously reducible to E_X.*

Proof. Let $\{S_\alpha \mid \alpha \in X\}$ be a sequence testifying $\diamondsuit_\kappa(X)$ and define the function $\mathcal{F} \colon 2^\kappa \to 2^\kappa$ by

$$\mathcal{F}(\eta)(\alpha) = \begin{cases} 1 & \text{if } \alpha \in X \cap C_\pi \cap C_{EF},\ \mathbf{II} \uparrow EF^\kappa_\omega(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_{S_\alpha}) \text{ and } \mathcal{A}_\eta \restriction_\alpha \models T \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that $\mathcal{F}$ is a reduction of $\cong_T$ to E_X, i.e. for every $\eta, \xi \in 2^\kappa$, $(\eta, \xi) \in\ \cong_T$ if and only if $(\mathcal{F}(\eta), \mathcal{F}(\xi)) \in E_X$. Notice that when $\alpha \in C_\pi$, the structure $\mathcal{A}_{\eta\restriction\alpha}$ is defined and equals $\mathcal{A}_\eta \restriction_\alpha$.

Consider first the direction from left to right. Suppose first that $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ are models of T and $\mathcal{A}_\eta \cong \mathcal{A}_\xi$. Since $\mathcal{A}_\eta \cong \mathcal{A}_\xi$, we have $\mathbf{II} \uparrow \text{EF}^\kappa_\omega(\mathcal{A}_\eta, \mathcal{A}_\xi)$. By Lemma 2.2 there is a club C such that $\mathbf{II} \uparrow \text{EF}^\alpha_\omega(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_\xi \restriction_\alpha)$ for every α in C. Since the set $\{\alpha < \kappa \mid \mathcal{A}_\eta \restriction_\alpha \models T, \mathcal{A}_\xi \restriction_\alpha \models T\}$ contains a club, we can assume that every $\alpha \in C$ satisfies $\mathcal{A}_\eta \restriction_\alpha \models T$ and $\mathcal{A}_\xi \restriction_\alpha \models T$. If $\alpha \in C$ is such that $\mathcal{F}(\eta)(\alpha) = 1$, then $\mathbf{II} \uparrow \text{EF}^\alpha_\omega(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_{S_\alpha})$. Since $\mathbf{II} \uparrow \text{EF}^\alpha_\omega(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_\xi \restriction_\alpha)$ and $\alpha \in C_{EF}$, we can conclude that $\mathbf{II} \uparrow \text{EF}^\alpha_\omega(\mathcal{A}_\xi \restriction_\alpha, \mathcal{A}_{S_\alpha})$. Therefore for every $\alpha \in C$, $\mathcal{F}(\eta)(\alpha) = 1$ implies $\mathcal{F}(\xi)(\alpha) = 1$. Using the same argument it can be shown that for every $\alpha \in C$, $\mathcal{F}(\xi)(\alpha) = 1$ implies $\mathcal{F}(\eta)(\alpha) = 1$. Therefore $\mathcal{F}(\eta)$ and $\mathcal{F}(\xi)$ coincide in a club and $(\mathcal{F}(\eta), \mathcal{F}(\xi)) \in E_X$.

Let us now look at the case where $(\eta, \xi) \in\ \cong_T$ and $\mathcal{A}_\eta$ is not a model of T (the case $T \not\models \mathcal{A}_\xi$ follows by symmetry). By the definition of $\cong_T$ we know that $\mathcal{A}_\xi$ is not a model of T either, so there is $\varphi \in T$ such that $\mathcal{A}_\eta \models \neg\varphi$ and $\mathcal{A}_\xi \models \neg\varphi$. Further, there is a club C such that for every $\alpha \in C$ we have $\mathcal{A}_\eta \restriction_\alpha \models \neg\varphi$ and $\mathcal{A}_\xi \restriction_\alpha \models \neg\varphi$. We conclude that for every $\alpha \in C$ we have that $\mathcal{A}_\eta \restriction_\alpha$ and $\mathcal{A}_\xi \restriction_\alpha$ are not models of T, and $\mathcal{F}(\eta)(\alpha) = \mathcal{F}(\xi)(\alpha) = 0$, so $(\mathcal{F}(\eta), \mathcal{F}(\xi)) \in E_X$.

Let us now look at the direction from right to left. Suppose first that $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ are models of T, and $\mathcal{A}_\eta \not\cong \mathcal{A}_\xi$.

By Remark 2, we know that $\mathbf{I} \uparrow \mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}_\eta, \mathcal{A}_\xi)$. By Lemma 2.2 there is a club C of α with

$$\mathbf{I} \uparrow \mathrm{EF}^{\alpha}_{\omega}(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_\xi \restriction_\alpha),$$

$\mathcal{A}_\xi \restriction_\alpha \models T$ and $\mathcal{A}_\eta \restriction_\alpha \models T$.

Since $\{\alpha \in X \mid \eta \cap \alpha = S_\alpha\}$ is stationary by the definition of $\diamondsuit_\kappa(X)$, also the set

$$\{\alpha \in X \mid \eta \cap \alpha = S_\alpha\} \cap C_\pi \cap C_{EF}$$

is stationary and every α in this set satisfies $\mathbf{II} \uparrow \mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_{S_\alpha})$. Therefore

$$C \cap \{\alpha \in X \mid \eta \cap \alpha = S_\alpha\} \cap C_\pi \cap C_{EF}$$

is stationary and a subset of $\mathcal{F}(\eta)^{-1}\{1\} \triangle \mathcal{F}(\xi)^{-1}\{1\}$, where $\triangle$ denotes the symmetric difference. We conclude that $(\mathcal{F}(\eta), \mathcal{F}(\xi)) \notin E_X$.

Let us finally assume that $(\eta, \xi) \notin \cong_T$ and $\mathcal{A}_\eta \not\models T$ (the case $\mathcal{A}_\xi \not\models T$ follows by symmetry). Assume towards a contradiction that $(\mathcal{F}(\eta), \mathcal{F}(\xi)) \in E^2_{\mu\text{-club}}$. Let C be a club that testifies $(\mathcal{F}(\eta), \mathcal{F}(\xi)) \in E^2_{\mu\text{-club}}$, i.e. $C \cap (\mathcal{F}(\eta)^{-1}[1] \triangle \mathcal{F}(\xi)^{-1}[1]) \cap X = \emptyset$. Since $\mathcal{A}_\eta \not\models T$, the set $\{\alpha < \kappa \mid \mathcal{A}_\eta \restriction_\alpha \not\models T\}$ contains a club. Hence, we can assume that for every $\alpha \in C$, $\mathcal{A}_\eta \restriction_\alpha \not\models T$ which implies that $\mathcal{F}(\eta)(\alpha) = 0$ and $\mathcal{F}(\xi)(\alpha) = 0$ for every $\alpha \in C$.

By the definition of $\cong_T$, $\mathcal{A}_\eta \not\models T$ implies $\mathcal{A}_\xi \models T$. Therefore the set $\{\alpha < \kappa \mid \mathcal{A}_\xi \restriction_\alpha \models T\}$ contains a club. So there is a club C' such that every $\alpha \in C'$ satisfies $\mathcal{A}_\xi \restriction_\alpha \models T$ and $\mathcal{F}(\xi)(\alpha) = 0$. Since $\{\alpha \in X \mid \xi \cap \alpha = S_\alpha\}$ is stationary, again by the definition of $\diamondsuit_\kappa(X)$, also $\{\alpha \in X \mid \eta \cap \alpha = S_\alpha\} \cap C_\pi \cap C_{EF}$ is stationary and every α in this set satisfies $\mathbf{II} \uparrow \mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_{S_\alpha})$. Therefore,

$$C' \cap \{\alpha \in X \mid \xi \cap \alpha = S_\alpha\} \cap C_\pi \cap C_{EF} \neq \emptyset,$$

a contradiction.

To show that $\mathcal{F}$ is continuous, let $[\eta \restriction_\alpha]$ be a basic open set, $\xi \in \mathcal{F}^{-1}[[\eta \restriction_\alpha]]$. Then $\xi \in [\xi \restriction_\alpha]$ and $[\xi \restriction_\alpha] \subseteq \mathcal{F}^{-1}[[\eta \restriction_\alpha]]$. We conclude that $\mathcal{F}$ is continuous. □

To define the reduction $\mathcal{F}$ it is not enough to use the isomorphism classes of the models $\mathcal{A}_{S_\alpha}$, as opposed to the equivalence classes of the relation defined by the EF-game. It is possible to construct two non-isomorphic models with domain κ such that their restrictions to any $\alpha < \kappa$ are isomorphic. For example the models $\mathcal{M} = (\kappa, P)$ and $\mathcal{N} = (\kappa, Q)$, with $\kappa = \lambda^+$,

$$P = \{\alpha < \kappa \mid \alpha = \beta + 2n,\ n \in \mathbb{N} \text{ and } \beta \text{ a limit ordinal}\}$$

and

$$Q = \{\alpha < \lambda \mid \alpha = \beta + 2n,\ n \in \mathbb{N} \text{ and } \beta \text{ a limit ordinal}\}$$

are non-isomorphic but $\mathcal{M} \restriction_\alpha \cong \mathcal{N} \restriction_\alpha$ holds for every $\alpha < \kappa$.

The Borel reducibility of the isomorphism relation of classifiable theories was studied in [FHK14] and one of the main results is the following.

Theorem 2.4. *([FHK14, Thm 77]) If a first order theory T is classifiable, then for all regular cardinals $\mu < \kappa$, $E^2_{\mu\text{-club}} \not\leq_B \cong^\kappa_T$.*

Corollary 2.5. *Assume that $\diamondsuit_\kappa(S^\kappa_\mu)$ holds for all regular $\mu < \kappa$. If a first order theory T is classifiable, then for all regular cardinals $\mu < \kappa$ we have $\cong^\kappa_T \leqslant_c E^2_{\mu\text{-club}}$ and $E^2_{\mu\text{-club}} \not\leq_B \cong^\kappa_T$.*

3 Non-classifiable Theories

In [FHK14] the reducibility to the isomorphism of non-classifiable theories was studied. In particular the following two theorems were proved there:

Theorem 3.1. *([FHK14, Thm 79]) Suppose that $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$.*

1. *If T is unstable or superstable with OTOP, then $E^2_{\lambda\text{-club}} \leqslant_c \cong^\kappa_T$.*
2. *If $\lambda \geqslant 2^\omega$ and T is superstable with DOP, then $E^2_{\lambda\text{-club}} \leqslant_c \cong^\kappa_T$.*

Theorem 3.2. *([FHK14, Thm 86]) Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and T is a stable unsuperstable theory. Then $E^2_{\omega\text{-club}} \leqslant_c \cong^\kappa_T$.*

Clearly from Theorems 3.1 and 3.2 and Corollary 2.3 we obtain the following:

Theorem 3.3. *Suppose that $\kappa = \lambda^+ = 2^\lambda$, $\lambda^{<\lambda} = \lambda$ and $\diamondsuit_\kappa(S^\kappa_\lambda)$ holds.*

1. *If T_1 is classifiable and T_2 is unstable or superstable with OTOP, then $\cong^\kappa_{T_1} \leqslant_c \cong^\kappa_{T_2}$ and $\cong^\kappa_{T_2} \not\leqslant_B \cong^\kappa_{T_1}$.*
2. *If $\lambda \geqslant 2^\omega$, T_1 is classifiable and T_2 is superstable with DOP, then $\cong^\kappa_{T_1} \leqslant_c \cong^\kappa_{T_2}$ and $\cong^\kappa_{T_2} \not\leqslant_B \cong^\kappa_{T_1}$.*

Theorem 3.4. *Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and $\diamondsuit_\kappa(S^\kappa_\omega)$ holds. If T_1 is classifiable and T_2 is stable unsuperstable, then $\cong^\kappa_{T_1} \leqslant_c \cong^\kappa_{T_2}$ and $\cong^\kappa_{T_2} \not\leqslant_B \cong^\kappa_{T_1}$.*

Corollary 3.5. *Suppose $\kappa = \kappa^{<\kappa} = \lambda^+$ and $\lambda^\omega = \lambda$. If T_1 is classifiable and T_2 is stable unsuperstable, then $\cong^\kappa_{T_1} \leqslant_c \cong^\kappa_{T_2}$ and $\cong^\kappa_{T_2} \not\leqslant_B \cong^\kappa_{T_1}$.*

Proof. In [She10] Shelah proved that if $\kappa = \lambda^+ = 2^\lambda$ and S is a stationary subset of $\{\alpha < \kappa \mid cf(\alpha) \neq cf(\lambda)\}$, then $\diamondsuit_\kappa(S)$ holds. Since $\lambda^\omega = \lambda$, we have $cf(\lambda) \neq \omega$ and $\diamondsuit_\kappa(S^\kappa_\omega)$ holds. On the other hand $\kappa = \lambda^+$ and $\lambda^\omega = \lambda$ implies $\gamma^\omega < \kappa$ for all $\gamma < \kappa$. By Theorem 3.4 we conclude that if T_1 is a classifiable theory and T_2 is a stable unsuperstable theory, then $\cong_{T_1} \leqslant_c \cong_{T_2}$ and $\cong_{T_2} \not\leqslant_B \cong_{T_1}$. □

Theorem 3.6. *Let $H(\kappa)$ be the following property: If T is classifiable and T' not, then $\cong^\kappa_T \leqslant_c \cong^\kappa_{T'}$ and $\cong^\kappa_{T'} \not\leqslant_B \cong^\kappa_T$. Suppose that $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$.*

1. *If $V = L$, then $H(\kappa)$ holds.*
2. *There is a κ-closed forcing notion $\mathbb{P}$ with the κ^+-c.c. which forces $H(\kappa)$.*

Proof.

1. This follows from Theorems 3.3 and 3.4.
2. Let $\mathbb{P}$ be $\{f \colon X \to 2 \mid X \subseteq \kappa, |X| < \kappa\}$ with the order $p \leqslant q$ if $q \subset p$. It is known that $\mathbb{P}$ has the κ^+-cc [Kun11, Lemma IV.7.5] and is κ-closed [Kun11, Lemma IV.7.14]. It is also known that $\mathbb{P}$ preserves cofinalities, cardinalities and subsets of κ of size less than κ [Kun11, Thm IV.7.9, Lemma IV.7.15]. Therefore, in $V[G]$, κ satisfies $\kappa = \kappa^{<\kappa} = \lambda^+ = 2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$. It is known that $\mathbb{P}$ satisfies $\mathbb{1} \Vdash_{\mathbb{P}} \diamondsuit_k(S^\kappa_\mu)$ for every regular cardinal $\mu < \kappa$. Therefore, by Theorems 3.3 and 3.4 $H(\kappa)$ holds in $V[G]$. □

Definition 3.7.

1. A tree T is a κ^+,κ*-tree* if does not contain chains of length κ and its cardinality is less than κ^+. It is *closed* if every chain has a unique supremum.

2. A pair (T,h) is a *Borel*-code* if T is a closed κ^+,κ-tree and h is a function with domain T such that if $x \in T$ is a leaf, then $h(x)$ is a basic open set and otherwise $h(x) \in \{\cup,\cap\}$.

3. For an element $\eta \in 2^\kappa$ and a *Borel**-code (T,h), the *Borel*-game* $B^*(T,h,\eta)$ is played as follows. There are two players, **I** and **II**. The game starts from the root of T. At each move, if the game is at node $x \in T$ and $h(x) = \cap$, then **I** chooses an immediate successor y of x and the game continues from this y. If $h(x) = \cup$, then **II** makes the choice. At limits the game continues from the (unique) supremum of the previous moves by Player **I**. Finally, if $h(x)$ is a basic open set, then the game ends, and **II** wins if and only if $\eta \in h(x)$.

4. A set $X \subseteq 2^\kappa$ is a *Borel*-set* if there is a *Borel**-code (T,h) such that for all $\eta \in 2^\kappa$, $\eta \in X$ if and only if **II** has a winning strategy in the game $B^*(T,h,\eta)$.

Note that a strategy in a game $B^*(T,h,\eta)$ can be seen as a function $\sigma : \kappa^{<\kappa} \to \kappa$, because every $\kappa^+\kappa$-tree can be seen as a downward closed subtree of $\kappa^{<\kappa}$.

Theorem 3.8. *Suppose that $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$. Then the following statements are consistent.*

1. *If T_1 is classifiable and T_2 is not, then there is an embedding of $(\mathcal{P}(\kappa),\subseteq)$ to $(B^*(T_1,T_2),\leqslant_B)$, where $B^*(T_1,T_2)$ is the set of all Borel*-equivalence relations strictly between $\cong_{T_1}$ and $\cong_{T_2}$.*

2. *If T_1 is classifiable and T_2 is unstable or superstable, then*

$$\cong^\kappa_{T_1} \leqslant_c E^2_{\lambda\text{-}club} \leqslant_c \cong^\kappa_{T_2} \wedge \cong^\kappa_{T_2} \not\leqslant_B E^2_{\lambda\text{-}club} \wedge E^2_{\lambda\text{-}club} \not\leqslant_B \cong^\kappa_{T_1} .$$

Proof. We will start the proof with two claims.

Claim 3.9. *If $\diamondsuit_\kappa(S)$ holds in V and $\mathbb{Q}$ is κ-closed, then $\diamondsuit_\kappa(S)$ holds in every $\mathbb{Q}$-generic extension.*

Proof. Let us proceed by contradiction. Suppose $(S_\alpha)_{\alpha\in S}$ is a $\diamondsuit_\kappa(S)$-sequence in V but not in $V[G]$, for some generic G. Fix the names $\check{S}, \dot{C}, \dot{X} \in V^{\mathbb{Q}}$ and $p \in G$, such that:

$$p \Vdash (\dot{C} \subseteq \check{\kappa} \text{ is a club } \wedge \dot{X} \subseteq \check{\kappa} \wedge \forall\alpha \in \dot{C}[\check{S}_\alpha \neq \dot{X}\cap\alpha]).$$

Working in V, we choose by recursion $p_\alpha, \beta_\alpha, \theta_\alpha$,and δ_α such that:

1. $p_\alpha \in \mathbb{Q}$, $p_0 = p$ and $p_\alpha \geqslant p_\gamma$ if $\alpha \leqslant \gamma$.
2. $\beta_\alpha \leqslant \beta_\gamma$ if $\alpha \leqslant \gamma$.
3. $\beta_\alpha \leqslant \theta_\alpha, \delta_\alpha < \beta_{\alpha+1}$.
4. If γ is a limit ordinal, then $\beta_\gamma = \delta_\gamma = \cup_{\alpha<\gamma}\beta_\alpha$.
5. $p_{\alpha+1} \Vdash (\check{\delta}_\alpha \in \check{C} \wedge \dot{X}\cap\check{\beta}_\alpha = \check{S}_{\theta_\alpha})$.

We will show how to choose them such that 1-5 are satisfied. First, for the successor step assume that for some $\alpha < \kappa$ we have chosen $p_{\alpha+1}, \beta_\alpha, \theta_\alpha$ and δ_α. We choose any ordinal satisfying 3 as $\beta_{\alpha+1}$. Since $p_{\alpha+1} \Vdash (\dot{C} \subseteq \check{\kappa}$ is a club), there exists $q \in \mathbb{Q}$ stronger than $p_{\alpha+1}$ and $\delta < \kappa$ such that $q \Vdash (\check{\delta} \in \dot{C} \wedge \check{\beta}_\alpha \leqslant \check{\delta})$. Now set $\delta_{\alpha+1} = \delta$. Since $\mathbb{Q}$ is κ-closed, there exists $Y \in \mathcal{P}(\beta_{\alpha+1})^V$ and $r \in \mathbb{Q}$ stronger than q such that $r \Vdash \dot{X}\cap\check{\beta}_{\alpha+1} = \check{Y}$. By $\diamondsuit_\kappa(S)$ in V, the set $\{\gamma < \kappa \mid Y = S_\gamma\}$ is stationary, so we can choose the least ordinal $\theta_{\alpha+1} \geqslant \beta_{\alpha+1}$ such that $r \Vdash \dot{X}\cap\check{\beta}_{\alpha+1} = \check{S}_{\theta_{\alpha+1}}$. Clearly $r = p_{\alpha+2}$ satisfies 1 and 5.

For the limit step, assume that for some limit ordinal $\alpha < \kappa$ we have chosen $p_\gamma, \beta_\gamma, \theta_\gamma$ and δ_γ for every $\gamma < \alpha$. Note that by 4 we know how to choose β_α and δ_α. Since $\mathbb{Q}$ is κ-closed, there exists p_α that satisfies 1. We choose θ_α as in the successor case with $q = p_\alpha$ and $p_{\alpha+1}$ as the condition r used to choose θ_α.

Define A, B and C_δ by $B = \cup_{\alpha<\kappa} S_{\theta_\alpha}$, $A = \{\alpha \in S \mid B \cap \alpha = S_\alpha\}$ and $C_\delta = \{\delta_\alpha \mid \alpha \text{ is a limit ordinal}\}$. Note that C_δ is a club. By $\diamondsuit_\kappa(S)$ in V, A is stationary and $A \cap C_\delta \neq \emptyset$. Let $\delta_\alpha \in A \cap C_\delta$. Then by 1, 2 and 5, for every $\gamma > \alpha$ we have $p_{\gamma+1} \Vdash (\check{S}_{\theta_\alpha} = \check{S}_{\theta_\gamma} \cap \check{\beta}_\alpha)$. Therefore, $S_{\theta_\alpha} = B \cap \beta_\alpha$ and $\delta_\alpha \in A \cap C_\delta$ and so by 4 we have $S_{\theta_\alpha} = B \cap \delta_\alpha = S_{\delta_\alpha}$. But now by 5 we get $p_{\alpha+1} \Vdash (\check{\delta}_\alpha \in \check{C} \wedge \dot{X} \cap \check{\delta}_\alpha = \check{S}_{\delta_\alpha})$ which is a contradiction. □

Claim 3.10. *For all stationary $X \subseteq \kappa$, the relation E_X is a Borel*-set.*

Proof. The idea is to code the club-game into the Borel*-game: in the club-game the players pick ordinals one after another and if the limit is in a predefined set A, then the second player wins. Define T_X as the tree whose elements are all the increasing elements of $\kappa^{\leqslant\lambda}$, ordered by end-extension. For every element of T_X that is not a leaf, define

$$H_X(x) = \begin{cases} \cup & \text{if } x \text{has an immediate predecessor } x^- \text{ and } H_X(x^-) = \cap \\ \cap & \text{otherwise} \end{cases}$$

and for every leaf b define $H_X(b)$ by:

$$(\eta, \xi) \in H_X(b) \iff \text{ for every } \alpha \in \lim(\mathrm{ran}(b)) \cap X(\eta(\alpha) = \xi(\alpha))$$

where $\alpha \in \lim(\mathrm{ran}(b))$ if $sup(\alpha \cap \mathrm{ran}(b)) = \alpha$.

Let us assume there is a winning strategy σ for Player **II** in the game $B^*(T_X, H_X, (\eta, \xi))$ and let us conclude that $(\eta, \xi) \in E_X$. Clearly by the definition of H_X we know that η and ξ coincide in the set $B = \{\alpha < \kappa \mid \sigma[dom(\sigma) \cap \alpha^{<\lambda}] \subset \alpha^{<\lambda}\} \cap X$. Since $\lambda^{<\lambda} = \lambda$, we know that $B' = \{\alpha < \kappa \mid \sigma[dom(\sigma) \cap \alpha^{<\lambda}] \subset \alpha^{<\lambda}\}$ is closed and unbounded. Therefore,there exists a club that doesn't intersect $(\eta^{-1}[1] \triangle \xi^{-1}[1]) \cap X$.

For the other direction, assume that $(\eta^{-1}[1] \triangle \xi^{-1}[1]) \cap X$ is not stationary and denote by C the club that does not intersect $(\eta^{-1}[1] \triangle \xi^{-1}[1]) \cap X$. The second player has a winning strategy for the game $B^*(T_\lambda, H_X, (\eta, \xi))$: she makes sure that, if b is the leaf in which the game ends and $A \subset \mathrm{ran}(b)$ is such that $\sup(\cup A) \in X$, then $\sup(\cup A) \in C$. This can be done by always choosing elements $f \in \kappa^{<\lambda}$ such that $\sup(\mathrm{ran}(f)) \in C$. □

Let $\mathbb{P}$ be $\{f \colon X \to 2 \mid X \subseteq \kappa, |X| < \kappa\}$ with the order $p \leqslant q$ if $q \subset p$. It is known that in any $\mathbb{P}$-generic extension, $V[G]$, $\diamondsuit_\kappa(S)$ holds for every $S \in V$, S a stationary subset of κ.

1. In [FHK14, Thm 52] the following was proved under the assumption $\kappa = \lambda^+$ and GCH:

   For every $\mu < \kappa$ there is a κ-closed forcing notion $\mathbb{Q}$ with the κ^+-c.c. which forces that there are stationary sets $K(A) \subsetneq S^\kappa_\mu$ for each $A \subsetneq \kappa$ such that $E_{K(A)} \not\leqslant_B E_{K(B)}$ if and only if $A \not\subset B$.

   In [FHK14, Thm 52] the proof starts by taking $(S_i)_{i<\kappa}$, κ pairwise disjoint stationary subsets of $lim(S^\kappa_\mu) = \{\alpha \in S^\kappa_\mu \mid \alpha \text{ is a limit ordinal in } S^\kappa_\mu\}$, and defining $K(A) = \cup_{\alpha \in A} S_\alpha$. $\mathbb{Q}$ is an iterated forcing that satisfies: For every name σ of a function $f : 2^\kappa \to 2^\kappa$, exists $\beta < \kappa$ such that, $\mathbb{P}_\beta \Vdash$ "*σ is not a reduction*".

   With a small modification on the iteration, it is possible to construct $\mathbb{Q}$ a κ-closed forcing with the κ^+-c.c. that forces

(∗) For $\mu \in \{\omega, \lambda\}$ and $A \subsetneq \kappa$, there are stationary sets $K(\mu, A) \subsetneq S^\kappa_\mu$ for which $E_{K(\mu,A)} \not\leqslant_B E_{K(\mu,B)}$ if and only if $A \not\subset B$.

Assume without loss of generality that GCH holds in V. Let G be a $\mathbb{P} * \mathbb{Q}$-generic. It is enough to prove that for every $A \subsetneq \kappa$ in $V[G]$ the following holds:

(a) If T_2 is unstable, or superstable with OTOP or with DOP, then $E_{K(\lambda,A)} \in B^*(T_1, T_2)$.

(b) If T_2 is stable unsuperstable, then $E_{K(\omega,A)} \in B^*(T_1, T_2)$.

In both cases the proof is the same; we will only consider (a).

Working in $V[G]$, let T_2 be as in (a). Since $\mathbb{Q}$ is κ-closed, we have $V[G] \models \diamondsuit_\kappa(S)$ for every stationary $S \subset \kappa$, $S \in V$. Since $\mathbb{P}$ and $\mathbb{Q}$ are κ-closed and have the κ^+-c.c., we have $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$. By Lemma 2.3, Theorems 3.1 and 3.4, we have that $\cong^\kappa_{T_1} \leqslant_c E_{K(\lambda,A)} \leqslant_c \cong^\kappa_{T_2}$ holds for every $A \subsetneq \kappa$. The argument in the proof of Theorem 2.4 can be used to prove that $E_{K(\lambda,A)} \not\leqslant_B \cong^\kappa_{T_1}$ holds for every $A \subsetneq \kappa$.

To show that $\cong^\kappa_{T_2} \not\leqslant_B E_{K(\lambda,A)}$ holds for every $A \subsetneq \kappa$, assume towards a contradiction that there exists $B \subsetneq \kappa$ such that $\cong^\kappa_{T_2} \leqslant_B E_{K(\lambda,B)}$. But then $E_{K(\lambda,A)} \leqslant_B E_{K(\lambda,B)}$ holds for every $A \subsetneq \kappa$ and by (∗), $A \subsetneq B$ for every $A \subsetneq \kappa$. So $B = \kappa$ which is a contradiction.

2. In [HK12, Thm 3.1] it is proved (under the assumptions $2^\kappa = \kappa^+$ and $\kappa = \kappa^{<\kappa} > \omega$) that there is a generic extension in which $\cong^\kappa_{DLO}$ is not a Borel*-set. The forcing is constructed using the following claim [HK12, Claim 3.1.5]:

   For each (t, h) there exists a κ^+-c.c. κ-closed forcing $\mathbb{R}(t, h)$ such that in any $\mathbb{R}(t, h)$-generic extension $\cong^\kappa_{DLO}$ is not a Borel-set.*

   The forcing in [HK12, Thm 3.1] works for every theory T that is unstable, or T non-classifiable and superstable (not only DLO, see [HK12] and [HT91]). Therefore, this claim can be generalized to:

   For each (t, h) there exists a κ^+-c.c. κ-closed forcing $\mathbb{R}(t, h)$ such that in any $\mathbb{R}(t, h)$-generic extension, $\cong^\kappa_T$ is not a Borel-set, for all T unstable, or T non-classifiable and superstable.*

   By iterating this forcing (as in [HK12, Thm 3.1]), we construct a forcing $\mathbb{Q}$ κ-closed, κ^+-c.c. that forces $\cong^\kappa_T$ *is not a Borel*-set*, for all T unstable, or T non-classifiable and superstable.

   Assume without loss of generality that $2^\kappa = \kappa^+$ holds in V. Let G be a $\mathbb{P} * \mathbb{Q}$-generic. Since $\mathbb{Q}$ is κ-closed, $V[G] \models \diamondsuit_\kappa(S)$ for every stationary $S \subset \kappa$, $S \in V$. Since $\mathbb{P}$ and $\mathbb{Q}$ are κ-closed and have the κ^+-c.c., we have $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$. Working in $V[G]$, let T_2 be unstable, or non-classifiable and superstable. By Lemma 2.3, Theorems 3.3 and 3.4 we finally have that $\cong^\kappa_{T_1} \leqslant_c E^2_{\lambda\text{-club}} \leqslant_c \cong^\kappa_{T_2}$ and $E^2_{\lambda\text{-club}} \not\leqslant_B \cong^\kappa_{T_1}$ holds.

   Since $2^\kappa \times 2^\kappa$ is homeomorphic to 2^κ, in order to finish the proof, it is enough to show that if $f\colon 2^\kappa \to 2^\kappa$ is Borel, then for all Borel*-sets A, the set $f^{-1}[A]$ is a Borel*. This is because if f were the reduction $\cong^\kappa_{T_2} \leqslant_B E^2_{\lambda\text{-club}}$, we would have $(f \times f)^{-1}[E^2_{\lambda\text{-club}}] = \cong^\kappa_{T_2}$ and since $E^2_{\lambda\text{-club}}$ is Borel*, this would yield the latter Borel* as well.

Claim 3.11. *Assume $f\colon 2^\kappa \to 2^\kappa$ is a Borel function and $B \subset 2^\kappa$ is Borel*. Then $f^{-1}[B]$ is Borel*.*

Proof. Let (T_B, H_B) be a Borel*-code for B. Define the Borel*-code (T_A, H_A) by letting $T_B = T_A$ and $H_A(b) = f^{-1}[H_B(b)]$ for every branch b of T_B. Let A be the Borel*-set coded by (T_A, H_A). Clearly, $\mathbf{II} \uparrow B^*(T_B, H_B, \eta)$ if and only if $\mathbf{II} \uparrow B^*(T_A, H_A, f^{-1}(\eta))$, so $f^{-1}[B] = A$. □

We end this paper with an open question:

Question 3.12. *Is it provable in ZFC that* $\cong^\kappa_T \lneq_B \cong^\kappa_{T'}$ *(note the strict inequality) for all complete first-order theories* T *and* T'*,* T *classifiable and* T' *not? How much can the cardinality assumptions on* κ *be relaxed?*

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