Connections between generalised Baire spaces and model theory

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Outline

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Borel and Δ_1^1 Sets

Definition

The collection of Borel subsets of κ^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ.

Definition

$A \subseteq \kappa^\kappa$ is an analytic if there is a closed subset F of the product space $\kappa^\kappa \times \kappa^\kappa$ such that its projection $pr(F) = \{\eta \in \kappa^\kappa \mid \exists \xi \in \kappa^\kappa\ (\eta, \xi) \in F\}$ is equal to A.

Definition

$A \subseteq \kappa^\kappa$ is a Δ_1^1 set if A and $\kappa^\kappa \backslash A$ are analytic sets.

Borel*-code

- A tree T is a κ^+, λ-*tree* if does not contain chains of length λ and its cardinality is less than κ^+. It is *closed* if every chain has a unique supremum.

- A pair (T, h) is a *Borel*-code* if T is a closed κ^+, κ-tree and h is a function with domain T such that if $x \in T$ is a leaf, then $h(x)$ is a basic open set and otherwise $h(x) \in \{\cup, \cap\}$.

Borel*-game

For an element $\eta \in \kappa^\kappa$ and a Borel*-code (T, h), the *Borel*-game* $B^*(T, h, \eta)$ is played as follows. There are two players, **I** and **II**. The game starts from the root of T.
At each move, if the game is at node $x \in T$ and $h(x) = \cap$, then **I** chooses an immediate successor y of x and the game continues from this y. If $h(x) = \cup$, then **II** makes the choice.
At limits the game continues from the (unique) supremum of the previous moves.
Finally, if $h(x)$ is a basic open set, then the game ends, and **II** wins if and only if $\eta \in h(x)$.

Borel* sets

Definition (Borel*)

A set $X \subseteq \kappa^\kappa$ is a Borel-set if there is a Borel*-code (T, h) such that for all $\eta \in \kappa^\kappa$, $\eta \in X$ if and only if* **II** *has a winning strategy in the game $B^*(T, h, \eta)$.*

We will write $\mathbf{II} \uparrow B^*(T, h, \eta)$ when **II** has a winning strategy in the game $B^*(T, h, \eta)$ and $\mathbf{I} \uparrow B^*(T, h, \eta)$ when **I** has a winning strategy in the game $B^*(T, h, \eta)$.

Definition (Dual sets)

We say that X and Y are duals if there is a Borel-code (T, h) such that:*

$$\eta \in X \Leftrightarrow \mathbf{II} \uparrow B^*(T, h, \eta),$$

$$\eta \in Y \Leftrightarrow \mathbf{I} \uparrow B^*(T, h, \eta),$$

Separation Theorem

Theorem (Mekler-Väänänen)

Suppose A and B are disjoint analytic sets. There are Borel sets C_0 and C_1 such that $A \subseteq C_0$ and $B \subseteq C_1$, and C_1 and C_0 are duals.*

Fact

X is a Borel set if and only if there is a Borel-code (T, h) coding X with T a κ^+, ω-tree.*

Corollary

- *X is Δ_1^1 if and only if there is a Borel*-code (T, h) coding X such that for all η*
$$\mathbf{II} \uparrow B^*(T, h, \eta) \Leftrightarrow \mathbf{I} \not\uparrow B^*(T, h, \eta).$$
- *Borel $\subseteq \Delta_1^1 \subseteq$ Borel*.*

Outline

1. Borel* sets
2. The isomorphism relation
3. Classifiable theories in the Borel hierarchy
4. The division line Classifiable vs Unclassifiable

The isomorphism relation

Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $\eta \in \kappa^\kappa$ define the structure $\mathcal{A}_\eta$ with domain κ and for every tuple $(a_1, a_2, \dots, a_n)$ in κ^n

$$(a_1, a_2, \dots, a_n) \in P_m^{\mathcal{A}_\eta} \Leftrightarrow \eta(\pi(m, a_1, a_2, \dots, a_n)) > 0$$

Definition

Given T a first-order complete countable theory in a countable vocabulary, we say that $\eta, \xi \in \kappa^\kappa$ are $\cong_T$ equivalent if

- $\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi$
  or
- $\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T$

Some Division Lines

- Unstable theories
- Stable unsuperstable theories
- Superstable theories with DOP
- Superstable theories with OTOP
- Superstable theories with no DOP nor OTOP

The tools used to construct models of a theory T, strongly depends on what kind of theory is T.

Ehrenfeucht-Fraïssé Game

Let $\mathcal{A}$ and $\mathcal{B}$ be structures with domain κ, and $\{X_\gamma\}_{\gamma<\kappa}$ an enumeration of the elements of $\mathcal{P}_\kappa(\kappa)$ and $\{f_\gamma\}_{\gamma<\kappa}$ an enumeration for all the functions with domain in $\mathcal{P}_\kappa(\kappa)$ and range in $\mathcal{P}_\kappa(\kappa)$. The game $EF^\kappa_\omega(\mathcal{A},\mathcal{B})$ is played by **I** and **II** as follows.

In the n-th turn **I** chooses an ordinal $\beta_n < \kappa$ such that $X_{\beta_{n-1}} \subset X_{\beta_n}$, and **II** an ordinal $\theta_n < \kappa$ such that $X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}} \subset f_{\theta_n}$, the game starts with X_{β_0} and f_{θ_0} as empty sets. The game finish after ω moves.

The player **II** wins if $\bigcup_{i<\omega} f_{\theta_i} : A \to B$ is a partial isomorphism, otherwise the player **I** wins.

Classifiable theories

A first-order complete countable theory in a countable vocabulary is classifiable if it is a superstable theory with no DOP nor OTOP.
We will write $\mathbf{II}\uparrow EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B})$ when $\mathbf{II}$ has a winning strategy in the game $EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B})$.

Fact

If T is a classifiable theory and $\mathcal{A}$, $\mathcal{B}$ are models of T, then

$$\mathbf{II}\uparrow EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \Leftrightarrow \mathcal{A}\cong\mathcal{B}.$$

Theorem (Friedman-Hyttinen-Kulikov)

If T is a classifiable theory, then $\cong_T$ is Δ^1_1. Moreover, if T is classifiable not shallow, then $\cong_T$ is not Borel.

Corollary

Borel $\neq \Delta^1_1$.

Outline

Reductions

A function $f\colon \kappa^\kappa \to \kappa^\kappa$ is *Borel*, if for every open set $A \subseteq \kappa^\kappa$ the inverse image $f^{-1}[A]$ is a Borel subset of κ^κ.

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

We write $E_1 \hookrightarrow_B E_2$ and we say that E_1 is as most as complex as E_2.

The Equivalence Modulo Non-stationary Ideals in GBS

Let $\lambda < \kappa$ be a regular cardinal. We say that $\eta, \xi \in \kappa^\kappa$ are $=_\lambda$ equivalent if the set $\{\alpha < \kappa | cof(\alpha) = \lambda \ \&\ \eta(\alpha) \neq \xi(\alpha)\}$ is not stationary.

Theorem (Hyttinen-M.)

Suppose T is a classifiable theory and $\lambda < \kappa$ is a regular cardinal. Then $\cong_T \ \hookrightarrow_B \ =_\lambda$.

Proof

For every $\alpha < \kappa$, structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the game $EF^\kappa_\omega(\mathcal{A}\restriction_\alpha, \mathcal{B}\restriction_\alpha)$ is played by **I** and **II** as follows.

In the *n*-th turn **I** chooses an ordinal $\beta_n < \alpha$ such that $X_{\beta_n} \subset \alpha$, $X_{\beta_{n-1}} \subset X_{\beta_n}$, and **II** an ordinal $\theta_n < \alpha$ such that $dom(f_{\theta_n}), rang(f_{\theta_n}) \subset \alpha$, $X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}} \subseteq f_{\theta_n}$.

The game starts with X_{β_0} and f_{θ_0} as empty sets, and finishes when one of the players cannot choose or after ω moves.

The player **II** wins if $\cup_{i<\omega} f_{\theta_i} : A\restriction_\alpha \to B\restriction_\alpha$ is a partial isomorphism, otherwise the player **I** wins.

Proof

Claim

For every pair of structures, $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the following holds:

- $\mathbf{II} \uparrow EF_\omega^\kappa(\mathcal{A}, \mathcal{B}) \iff \mathbf{II} \uparrow EF_\omega^\kappa(\mathcal{A} \restriction_\alpha, \mathcal{B} \restriction_\alpha)$ *for club-many α.*
- $\mathbf{I} \uparrow EF_\omega^\kappa(\mathcal{A}, \mathcal{B}) \iff \mathbf{I} \uparrow EF_\omega^\kappa(\mathcal{A} \restriction_\alpha, \mathcal{B} \restriction_\alpha)$ *for club-many α.*

Definition

Given T a first-order complete countable theory in a countable vocabulary and $\alpha \leqslant \kappa$, define the relation $R_{EF}^\alpha \subseteq \kappa^\kappa \times \kappa^\kappa$ as $\eta\ R_{EF}^\alpha\ \xi$:

- $\mathcal{A}_\eta \restriction_\alpha \not\models T$ *and* $\mathcal{A}_\xi \restriction_\alpha \not\models T$*, or*
- $\mathcal{A}_\eta \restriction_\alpha \models T$, $\mathcal{A}_\xi \restriction_\alpha \models T$ *and the player* $\mathbf{II}$ *has a winning strategy for the restricted game* $EF_\omega^\kappa(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_\xi \restriction_\alpha)$.

Proof

Claim

For every T first-order complete countable theory in a countable vocabulary, there are club many α such that R^{α}_{EF} is an equivalence relation.

Define the reduction as follows.
For every $\eta \in \kappa^\kappa$ define the function f_η, as:

- $f_\eta(\alpha)$ is a code in $\kappa\backslash\{0\}$ for the R^{α}_{EF} equivalence class for $\mathcal{A}_\eta\restriction_\alpha$, when $cf(\alpha)=\lambda$, $\mathcal{A}_\eta\restriction_\alpha\models T$, and R^{α}_{EF} is an equivalence relation;
- $f_\eta(\alpha)=0$ in other case.

The Cantor Space

The isomorphism relation and the equivalence modulo non-stationary ideals can be easily define in the generalised Cantor space 2^κ.

$$\cong^2_T \text{ is } \cong_T \cap (2^\kappa \times 2^\kappa)$$
$$=^2_\lambda \text{ is } =_\lambda \cap (2^\kappa \times 2^\kappa)$$

Theorem (Hyttinen-Kulikov-M.)

*Denote by S^κ_λ the set $\{\alpha < \kappa | cf(\alpha) = \lambda\}$.
Suppose T is a classifiable theory and $\lambda < \kappa$ is a regular cardinal. If $\diamondsuit(S^\kappa_\lambda)$ holds, then $\cong^2_T \ \hookrightarrow_B \ =^2_\lambda$.*

Outline

Unstable

Theorem (Friedman-Hyttinen-Kulikov)

If T is unstable, then $\cong_T$ is not Δ_1^1.

Theorem (Friedman-Hyttinen-Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$. If T is an unstable, then $=^2_\lambda \ \hookrightarrow_B \ \cong^2_T$.

Theorem (Hyttinen-Kulikov-M.)

Suppose $\kappa = \lambda^+ = 2^\lambda$, $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(S^\kappa_\lambda)$ holds. If T is a classifiable theory and T' is an unstable, then $\cong^2_T \ \hookrightarrow_B \ =^2_\lambda \ \hookrightarrow_B \ \cong^2_{T'}$.

Stable unsuperstable

Question

Can it be proved in ZFC that if T is stable unsuperstable, then $\cong_T$ is not Δ^1_1?

Theorem (Friedman-Hyttinen-Kulikov)

Suppose for all $\lambda < \kappa$, $\lambda^\omega < \kappa$. If T is a stable unsuperstable, then $=^2_\omega \ \hookrightarrow_B \ \cong^2_T$.

Theorem (Hyttinen-Kulikov-M.)

Suppose $\kappa = \lambda^+$ and $\lambda^\omega = \lambda$. If T is a classifiable theory and T' is a stable unsuperstable theory, then $\cong^2_T \ \hookrightarrow_B \ =^2_\omega \ \hookrightarrow_B \ \cong^2_{T'}$ and $\cong^2_{T'} \ \not\hookrightarrow_B \ \cong^2_T$.

The Orthogonal Chain Property (OCP)

Definition

Given $p \in S(A)$ and $B \subseteq A$, we say $p \perp B$ if for every $q \in S(A)$ that doesn't fork over B the following holds; for every a, b, and $B' \supseteq A$, if a realizes p, b realizes q, $a \downarrow_A B'$ and $b \downarrow_A B'$, then $a \downarrow_{B'} b$.

Definition

A stable theory T has the property OCP if there exist $\lambda_r(T)$-saturated models of T of power $\lambda_r(T)$, $\{\mathcal{A}_i\}_{i<\omega}$, and $a \notin \cup_{i<\omega}\mathcal{A}_i$ such that for all $i \leqslant j$, $\mathcal{A}_i \subseteq \mathcal{A}_j$, $t(a, \cup_{i<\omega}\mathcal{A}_i)$ is not algebraic and for all $j < \omega$, $t(a, \cup_{i<\omega}\mathcal{A}_i) \perp \mathcal{A}_j$.

Theorem (Hyttinen-M.)

Suppose T is a classifiable theory, T' is a stable theory with the OCP, and κ is an inaccessible cardinal. Then $\cong_T \ \hookrightarrow_B \ =_\omega \ \hookrightarrow_B \ \cong_{T'}$

Superstable with DOP

Definition (a-isolation)

Denote by F^a_ω the set of pairs (p, A) with $|A| < \omega$, such that for some $B \supseteq A$, $p \in S(B)$, $a \models p$ and $stp(a, A) \vdash p$.

From this isolation the notions of a-saturated, a-primary, and a-minimal are defined.

Definition (DOP)

A theory T has the dimensional order property if there are a-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, $M_1 \downarrow_{M_0} M_2$, and the a-primary model over $M_1 \cup M_2$ is not a-minimal over $M_1 \cup M_2$.

Superstable with DOP

Theorem (Friedman-Hyttinen-Kulikov)

If T is superstable with DOP and $\kappa > \omega_1$, then $\cong_T$ is not Δ_1^1.

Theorem (Friedman-Hyttinen-Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda > 2^\omega$. If T is superstable with DOP, then $=^2_\lambda \ \hookrightarrow_B \ \cong^2_T$.

Theorem (Hyttinen-Kulikov-M.)

Suppose $\kappa = \lambda^+ = 2^\lambda$, $\lambda^{<\lambda} = \lambda > 2^\omega$ and $\diamondsuit(S^\kappa_\lambda)$ holds. If T is a classifiable theory and T' is superstable with DOP, then $\cong^2_T \ \hookrightarrow_B \ =^2_\lambda \ \hookrightarrow_B \ \cong^2_{T'}$.

Superstable with S-DOP

Definition (S-DOP)

We say that a theory T has the strong dimensional order property if the following holds:
There are a-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, such that
$M_1 \downarrow_{M_0} M_2$
, and for every M_3 a-primary model over $M_1 \cup M_2$, there is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, that does not fork over $M_1 \cup M_2$.

Theorem (M.)

Suppose T is a classifiable theory, T' is a superstable theory with the S-DOP, $\lambda = (2^\omega)^+$, and κ an inaccessible cardinal. Then
$\cong_T \quad \hookrightarrow_B \quad =_\lambda \quad \hookrightarrow_B \quad \cong_{T'}$

Superstable with OTOP

Theorem (Friedman-Hyttinen-Kulikov)

If T is stable with OTOP, then $\cong_T$ is not Δ_1^1.

Theorem (Friedman-Hyttinen-Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$. If T is superstable with OTOP, then $=^2_\lambda \ \hookrightarrow_B \ \cong^2_T$.

Theorem (Hyttinen-Kulikov-M.)

Suppose $\kappa = \lambda^+ = 2^\lambda$, $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(S^\kappa_\lambda)$ holds. If T is a classifiable theory and T' is superstable with OTOP, then $\cong^2_T \ \hookrightarrow_B \ =^2_\lambda \ \hookrightarrow_B \ \cong^2_{T'}$.

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

- By the Borel reducibility hierarchy.

  Let $H(\kappa)$ be the following property: If T is a classifiable theory and T' is not a classifiable theory, then $\cong^2_T \ \hookrightarrow_B \ \cong^2_{T'}$ and $\cong^2_{T'} \ \not\hookrightarrow_B \ \cong^2_T$.

Theorem (Hyttinen-Kulikov-M.)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$.

1. *If $\diamondsuit(S^\kappa_\omega)$ and $\diamondsuit(S^\kappa_\lambda)$ hold, then $H(\kappa)$ holds.*
2. *It is consistent that $H(\kappa)$ holds and there are 2^κ equivalence relations strictly between $\cong^2_T$ and $\cong^2_{T'}$.*

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

- By the different analytic sets.

Theorem (Friedman-Hyttinen-Kulikov)

- *If T is classifiable and Shallow, then $\cong_T$ is Borel.*
- *If T is classifiable but not shallow, then $\cong_T$ is Δ_1^1 but not Borel.*
- *If T is stable unsuperstable, then $\cong_T$ is analytic but not Borel.*
- *If T is unstable or stable with OTOP, then $\cong_T$ is analytic but not Δ_1^1.*
- *If T is superstable with DOP and $\kappa > \omega_1$, then $\cong_T$ is analytic but not Δ_1^1.*

Sum up

By using both approaches at the same time:

Let $G(\kappa)$ be the following property: If T is a countable first-order theory in a countable vocabulary, not necessarily complete, then one of the following holds:

- $\cong_T$ is Δ^1_1.
- $\cong_T$ is Σ^1_1-complete.

Theorem (Hyttinen-Kulikov-M.)

Suppose $V = L$. $G(\kappa)$ holds for all κ successor of a regular uncountable cardinal λ.

Theorem (Fernandes-M.-Rinot)

Suppose $\kappa = \lambda^+$ and $\lambda^{<\lambda} = \lambda > \omega$. There exists a $< \kappa$-closed κ^+-cc forcing extension, in which $G(\kappa)$ holds.

Fake reflection

It is a consequence of Π^1_1-filter reflection with $\diamondsuit$, $Dl^*_S(\Pi^1_1)$.

Definition (Filter reflection with $\diamondsuit$)

Suppose X and S are stationary subsets of κ, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α.

1. *We say that $\vec{\mathcal{F}}$ captures clubs iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is nonstationary;*
2. *We say that X $\vec{\mathcal{F}}$-reflects with $\diamondsuit$ to S iff $\vec{\mathcal{F}}$ captures clubs and there exists a sequence $\langle Y_\alpha \mid \alpha \in S \rangle$ such that, for every stationary $Y \subset X$, the set $\{\alpha \in S \mid Y_\alpha = Y \cap \alpha \ \&\ Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ is stationary.*

It can be forced by Sakai's forcing, Friedman-Holy's forcing, and Holy-Welch-Wu's forcing. It can be killed by $\mathrm{Add}(\kappa, \kappa^+)$. It follows from $V = L$ but also from Martin's Maximum.

Questions

Question

Is Δ_1^1 =Borel* consistently true?

Question

Can it be proved in ZFC that if T is stable unsuperstable, then $\cong_T$ is not Δ_1^1?

Question

Ssuppose T is a first-order countable complete theory over a countable vocabulary. If T is stable unsuperstable, then T has the OCP?

Thank you

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